

Some useful counterexamples regarding comonotonicity

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Abstract. This article gives counterexamples for some conjectures about risk orders. One is that in risky situations, diversification is always beneficial. A counterexample is provided by the Cauchy distribution, for which the sample means have the same distribution as the sample elements, meaning that insuring half the sum of two iid risks of this type is precisely equivalent to insuring one of them fully. In this case, independence and comonotonicity for these two risks are equivalent. We also show that if X, Y are iid $\text{Pareto}(\alpha, 1)$ with $\alpha < 1$, for the values-at-risk we have $F_{X+Y}^{-1}(q) > F_{2X}^{-1}(q)$ for q large enough. This proves that a sum of iid risks might be worse than a sum of corresponding comonotonic risks in the sense of having lower values-at-risk in the far-right tail. Then comonotonicity is preferable to independence, so independence is certainly not a ‘worst case’ scenario. Finally we show that if one risk has smaller stop-loss premiums than another, this doesn’t have to mean that its cdf is above the other from a certain point on. We give an example that the sum of independent risks can have a cdf that crosses infinitely often with its comonotonic equivalent. That such a distribution exists is no surprise, but an example has never been exhibited so far.

Samenvatting. Dit artikel geeft tegenvoorbeelden voor een aantal vermoedens die er leven over risico-ordeningen. Een daarvan is dat in geval van risico’s, het onveranderlijk voordelig is om te diversificeren. Een tegenvoorbeeld hiervan levert de Cauchy-verdeling, waarbij het steekproefgemiddelde en de steekproefelementen dezelfde verdeling hebben. Dit houdt in dat het verzekeren van de helft van de som van twee o.o.i. risico’s precies equivalent is aan het volledig verzekeren van één ervan. In dit geval zijn comonotonie en onafhankelijkheid voor deze twee risico’s equivalent. We laten ook zien dat als X, Y o.o.i. $\text{Pareto}(\alpha, 1)$ kansveranderlijken zijn en $\alpha < 1$, voor de VaR’s ervan er geldt dat $F_{X+Y}^{-1}(q) > F_{2X}^{-1}(q)$ voor q groot genoeg. Dit laat zien dat een som van o.o.i. risico’s onaantrekkelijker kan zijn dan een som van overeenkomstige comonotone risico’s in de zin dat de laatste, diep in de rechterstaart van de verdeling, kleinere VaR’s hebben. Dan is comonotonie te verkiezen boven onafhankelijkheid, en is onafhankelijkheid geenszins een ‘worst case’ scenario. Ten slotte tonen we aan dat als een risico kleinere stop-loss premies heeft dan een ander risico, dit niet hoeft te betekenen dat de vdf ervan vanaf een zeker punt boven de andere ligt. We geven een voorbeeld dat de vdf van de som van onafhankelijke risico’s oneindig vaak kruist met die van het

comonotone equivalent ervan. Dat zo’n vdf bestaat zal niemand verbazen, maar tot op heden is er nooit een tegenvoorbeeld beschreven.

Keywords: Risk orders, Comonotonicity, Value-at-risk, Subexponential distributions

Sleutelwoorden: Ordenen van risico’s, Comonotonie, Value-at-Risk, Subexponentiële verdelingen

1 Introduction

When one considers sums of random variables of which the marginal cdf’s are known but the dependency structure is not, it follows from e.g. Chapter 10.6 of Kaas *et al.* (2001) that the riskiest choice is to take the random variables as dependent as possible, i.e., comonotonic. Indeed, let X and Y be random variables with cdf’s F and G respectively, and define $X^c = F^{-1}(U)$ and $Y^c = G^{-1}(U)$ for some U which is $\text{Uniform}(0, 1)$ distributed. The random variables X^c and Y^c have the same marginal distributions as X and Y , and they are comonotonic, in the sense that the support of (X^c, Y^c) is a set of which the elements are ordered componentwise. When $d_1 + d_2 = d$ for some (d_1, d_2) in the support of (X^c, Y^c) , we have $(X^c - d_1)_+ + (Y^c - d_2)_+ \equiv (X^c + Y^c - d)_+$, and since $(X + Y - d)_+ \leq (X - d_1)_+ + (Y - d_2)_+$ a.s., if X and Y have finite mean we have $E[(X + Y - d)_+] \leq E[(X^c + Y^c - d)_+]$ for their stop-loss premiums. Therefore, we have $X + Y \leq_{cx} X^c + Y^c$, where the symbol \leq_{cx} (convex order) denotes lower stop-loss premiums and the same mean. Convex smaller risks are more attractive for risk-averse decision makers, having, e.g., a smaller variance. Especially, when X and Y are iid we have $X + Y \leq_{cx} 2X$. The convex order concept requires finite stop-loss premiums, hence *finite means*. For a more thorough treatment of comonotonicity and the state-of-the-art of the theory, consult Dhaene *et al.* (2002a,b).

Apparently comonotonicity is the ‘worst case’ scenario if the means are finite, but if not, unexpected things may happen. For instance, diversification does not always improve the risk situation. It is quite easy to give a situation that a sample average, which is a special case of diversification with iid risks, is just as bad as each sample element. But it may also happen that it is actually worse, in the sense that the values-at-risk of a sum of independent random variables exceed those of a comonotonic sum. So it is sometimes better to put all ones eggs in one basket. Finally, using the techniques of heavy-tailed distributions, see Embrechts *et al.* (1997), we give an example of a cdf with a finite mean which crosses infinitely often with the cdf of its sample mean, disproving the conjec-

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ture that two convex ordered cdf's must be ordered from a certain point on. This will not come as a surprise to the initiated, but an actual example, involving a rather natural random variable, has so far not been published.

2 Counterexamples

2.1 Cauchy distribution

One of the cornerstones of insurance is that the variation coefficient of the total risk diminishes if additional independent risks are insured. We will show that this is not always the case, simply by noting an example in which the sample mean is just as unattractive as each sample element. This means that diversification of risks has then proved to be pointless. For this, we look at the Cauchy distribution, which is in fact a Student's t_1 distribution, obtained by taking the ratio of two iid $N(0, 1)$ random variables. Admittedly, the Cauchy distribution is not an obvious choice to model an insurance risk. But as is well-known in probability and statistics, this distribution, with density

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty,$$

has the peculiar property that the sample mean has the same distribution as each sample element, hence

$$\bar{X}_n = \frac{X_1 + \dots + X_n}{n} =_d X_1. \quad (1)$$

See, e.g., Feller (1971), page 51. So this distribution may serve as a counterexample for any illusion one might have that a sample average is *always* 'better' than each sample element, either as an estimator or as a loss for risk-averse decision makers. If n insurance companies carry iid risks with this distribution, risk pooling (with everyone carrying an equal share of the total risk) fails to improve the situation for anyone.

2.2 Pareto distribution

Making the tail of the distribution even heavier, we are able to give examples in which for the values-at-risk, we have $F_{X+Y}^{-1}(q) > F_{X^c+Y^c}^{-1}(q)$ for q large enough. Therefore, the sum of two independent risks can be more dangerous than the sum of their comonotonic equivalents in the sense that ultimately, its values-at-risk are smaller.

Consider iid random variables X and Y with a common cdf F satisfying for some $\alpha > 0$:

$$\bar{F}(x) = 1 - F(x) = (1+x)^{-\alpha}, \quad x > 0. \quad (2)$$

Thus, $X+1$ has a Pareto($\alpha, 1$) distribution. Note that for $\alpha = 1$ the density equals $(1+x)^{-2}$, closely resembling the Cauchy density of the previous example. To make stronger statements than in the previous example, namely that sample averages are actually 'worse' than sample elements, we look at smaller α -values. For this distribution, for any $0 < \varepsilon < 1$ and any positive α such that $2^{1-\alpha} > 1 + \varepsilon$ holds, the inequality

$$\Pr[X+Y > x] > (1+\varepsilon) \Pr[X^c+Y^c > x] \quad (3)$$

holds for all large x , say $x > M = M(\varepsilon, \alpha)$, where (X^c, Y^c) is the comonotonic version of (X, Y) . The simplest way to prove this is to observe that by a straightforward application of Lemma 1.3.1 of Embrechts *et al.* (1997) or the Proposition on page 278 in Feller (1971), we have

$$\lim_{x \rightarrow \infty} \frac{\Pr[X+Y > x]}{\Pr[X^c+Y^c > x]} = 2^{-\alpha+1} > 1 + \varepsilon.$$

From (3) we see that for large enough x , the values-at-risk for $X+Y$ with independent X and Y are larger than those with their comonotonic counterparts.

2.3 A cdf crossing infinitely often with the one of its sample mean

For two risks with the same mean to be convex-ordered, it suffices that their cdf's cross once, with the smaller risk having the larger cdf in the right tail. Of course we cannot conclude that when one risk has smaller stop-loss premiums than another, this will show because its tail probabilities, and its values-at-risk, are uniformly lower from a certain point on. Though this is the case in many situations, as Taylor (2002) surmised it is possible to construct counterexamples. We show in fact that not even the existence of m moments is sufficient for this property: for any $m = 1, 2, \dots$, we can exhibit a distribution with finite m th moment, such that if X and Y are iid with this cdf and therefore $X+Y \leq_{cx} 2X$, to the right of any $M > 0$ the cdf's F_{X+Y} and F_{2X} cross infinitely often. We prove this using well-known results from the theory of heavy-tailed distributions. This example also provides a negative answer to the following question: for two random variables X and Y with a finite mean, does it always hold that for all sufficiently large $x > 0$, we have $\Pr[X+Y > x] \leq \Pr[X^c+Y^c > x]$?

Let X and Y be two iid random variables, and suppose X is given by

$$X = (1+U)2^{2N+B}, \quad (4)$$

where U , N and B are independent random variables, with U uniform(0, 1), N geometric with $\Pr[N = k] = (1-p)p^k$ for $k = 0, 1, \dots$, and B Bernoulli($\frac{(a-1)p}{1-p}$). Then X and Y have the following continuous marginal cdf F , with support $[1, \infty)$:

$$\bar{F}(x) = \begin{cases} p^k & \text{if } x = 2^{2k}, \\ ap^{k+1} & \text{if } x = 2^{2k+1}, \\ \text{linear} & \text{otherwise.} \end{cases} \quad k = 0, 1, \dots, \quad (5)$$

By the factor $1+U$ in X , F has a density which is piecewise constant, and it is oscillating because of the factor 2^B . The cdf F is well-defined as long as $0 < p < 1$ and $1 < a < \frac{1}{p}$, but from now on let's further assume that $0 < \varepsilon < \frac{2}{a} - 1$ holds, hence $a < 2$, and that for some integer $m = 1, 2, \dots$, we have $0 < p < 4^{-m}$. Then the m th moment of X can be seen to be finite. It is easy to check that the cdf F defined by (5) is long-tailed ($\lim_{x \rightarrow \infty} \bar{F}(x+1)/\bar{F}(x) = 1$) and has a

dominatedly varying tail ($\limsup_{x \rightarrow \infty} \overline{F}(x)/\overline{F}(2x) < \infty$). From Embrechts *et al.* (1997), Chapter 1.4.1, we immediately conclude that F is subexponential, and hence for every $n = 1, 2, \dots$,

$$\lim_{x \rightarrow \infty} \frac{\overline{F}^{*n}(x)}{\Pr[\text{Max}\{X_1, \dots, X_n\} > x]} = 1. \quad (6)$$

Especially for the case $n = 2$, noting that $\Pr[\text{Max}\{X, Y\} > x] = 1 - (1 - \overline{F}(x))^2 = 2\overline{F}(x) - (\overline{F}(x))^2$, we have the following result for the convolution F^{*2} :

$$\lim_{x \rightarrow \infty} \frac{\Pr[X + Y > x]}{\Pr[X > x]} = 2.$$

Note that this can be interpreted as $\Pr[X + Y > x] \approx \Pr[X > x] + \Pr[Y > x]$ for large x . One might say that $X + Y$ is ‘catastrophic’ (i.e., $> x$) if X or Y is, and only with vanishing probability if both lead to a partial catastrophe.

Let X^c and Y^c be comonotonic copies of X and Y , then $X^c + Y^c \equiv 2X^c$. Therefore

$$\lim_{k \rightarrow \infty} \frac{\Pr[X + Y > 2^{2k}]}{\Pr[X^c + Y^c > 2^{2k}]} = \lim_{k \rightarrow \infty} \frac{2\overline{F}(2^{2k})}{\overline{F}(2^{2k-1})} = \frac{2}{a}. \quad (7)$$

On the other hand, we also have

$$\lim_{k \rightarrow \infty} \frac{\Pr[X + Y > 2^{2k-1}]}{\Pr[X^c + Y^c > 2^{2k-1}]} = \lim_{k \rightarrow \infty} \frac{2\overline{F}(2^{2k-1})}{\overline{F}(2^{2k-2})} = 2ap. \quad (8)$$

Apparently,

$$\lim_{x \rightarrow \infty} \frac{\Pr[X + Y > x]}{\Pr[X^c + Y^c > x]}$$

does not exist since by our assumptions, $2/a > 1$ and $2ap < 1$.

We conclude from (7) that the inequality

$$\Pr[X + Y > x] > (1 + \varepsilon) \Pr[X^c + Y^c > x] \quad (9)$$

holds for any $0 < \varepsilon < \frac{2}{a} - 1$ and all $x = 2^{2k}$ for large k , hence on an unbounded set of points. So we have found a distribution with finite mean (in fact, m finite moments), such that if X and Y are iid with this cdf, and therefore $X + Y \leq_{cx} 2X$, to the right of any $M > 0$ the cdf’s F_{X+Y} and F_{2X} cross infinitely often.

Our example (4) also has its uses in various other situations.

Remark 2.1. For a cdf F with support $(0, \infty)$, the intermediate regularity property,

$$\lim_{y \searrow 1} \liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1, \quad (10)$$

introduced by Cline (1994), has been applied to investigations on precise large deviations, queueing systems and ruin theory. A question that has been frequently asked in the literature is whether this property is really weaker than the well-known regularity property,

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha} \quad \forall y > 0, \quad (11)$$

for some $\alpha \geq 0$. For the random variable X in (4), it is not difficult to prove that its cdf does not satisfy (11) for any $\alpha \geq 0$ but satisfies (10), so it is not regular but it is intermediate regular. This example is simpler and more intuitive than the one given by Cline and Samorodnitsky (1994) on page 87. An even simpler example is the random variable $(1 + U)2^N$.

Remark 2.2. As pointed out by Konstantinides *et al.* (2002), in deriving an asymptotic formula for the ultimate ruin probability of the classical model with a positive constant interest force, besides the subexponentiality an additional restriction,

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < 1 \quad \forall y > 1, \quad (12)$$

plays a crucial role. This fact was overlooked in some previous references in the literature. The question arises if there is any subexponential distribution of interest which does not satisfy the restriction (12) and does not have a slowly varying tail. The cdf of the random variable $Z = (1 + U)2^{2N}$ with U and N as in (4) can act as such an example. It is subexponential just as X , it does not have a slowly varying tail since

$$\frac{\Pr[Z > 4 \cdot 4^k]}{\Pr[Z > 4^k]} = \frac{p^{k+1}}{p^k} = p \quad \forall k = 0, 1, \dots,$$

so this ratio does not tend to 1, but on the other hand

$$\limsup_{x \rightarrow \infty} \frac{\Pr[Z > 2x]}{\Pr[Z > x]} = 1.$$

It is interesting to note that, just as above, for any positive integer m we may choose $0 < p < 1$ such that the m th moment of the random variable Z is finite. Hence this random variable is not very ‘heavy-tailed’.

3 Conclusion

As the examples above show, our intuition sometimes fails us in case of properties of less or more risky situations. The examples given, and variants of them, may serve to outline the boundaries of what actually holds.

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