Worst case risk measurement: back to the future?*

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Abstract

This paper studies the problem of finding best-possible upper bounds on a rich class of risk measures, expressible as integrals with respect to measures, under incomplete probabilistic information. Both univariate and multivariate risk measurement problems are considered. The extremal probability distributions, generating the worst case scenarios, are also identified.

The problem of worst case risk measurement has been studied extensively by Etienne De Vijlder and his co-authors, within the framework of finite-dimensional convex analysis. This paper revisits and extends some of their results.

Keywords: Risk measurement, Generalized scenarios, Worst case scenario, Cones, Linear programming, Value-at-Risk, Tail-Value-at-Risk, Exponential premium

JEL-Classification: D81, G10, G20

MSC-Classification: 60E05, 62P05, 90C05

*In honor of Etienne De Vijlder. Etienne passed away in 2004. To honor his large contribution to various fields within Actuarial Science — not limited to the type of problems discussed in this paper — we dedicate this work to him.

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1 Introduction

This paper studies the problem of finding best-possible upper bounds on risk measures when there is incomplete probabilistic information on the risks under consideration. Traditionally, the measurement of *univariate* risk under incomplete probabilistic information has been a main problem in non-life insurance. More recently, the measurement of *multivariate* risk under incomplete probabilistic information has become increasingly important. Developments in the insurance and financial industry, such as the (current and upcoming) solvency capital accords and the explosive growth in multi-name financial derivative products, urge for appropriate techniques for the measurement of multivariate risk (Genest *et al.*, 2009).

A main problem for the measurement of risk is that in practice often (only) partial information is available on the risks under study. A sound approach in that case is to investigate conditional worst case scenarios: given characteristics that only partially describe the risks, one searches for the most adverse scenario. This is also a good strategy for stress testing. For example, in a multivariate setting with a lack of information on the level of dependence between several risks, one could assume marginal distributions to be known, and identify the most adverse dependence scenario, conducting a stress test for dependence.

In this paper, we will mainly be concerned with a rich class of risk measures, namely risk measures that are expressible as integrals with respect to measures. The availability of partial probabilistic information is formalized by imposing integral constraints, and conical constraints on the cone of measures under consideration. We study both univariate and multivariate risk measurement problems in this setting.

In the financial mathematics literature, the elements of (possibly degenerate) sets of measures are sometimes called *generalized scenarios*, and the worst case measure generating the best-possible upper bound on a risk measure is often referred to as the *worst case scenario*. In general, according to Wald (1950), worst case risk measurement, that is, using a maxmin decision criterion, seems reasonable when an a priori probability measure does not exist or is unknown to the decision maker; see also Huber (1981) and Gilboa & Schmeidler (1989) for references in this direction. For policy making purposes, worst case risk measurement should not be viewed as a substitute for Bayesian decision making à la Savage (1954), but rather as a way of constructing and assessing a prior; see also the discussion in Sims (2001).

The problem of finding best-possible upper bounds on integrals with respect to measures has been studied extensively by the late Etienne De Vijlder and his co-authors; see

De Vylder[1](1982, 1983a,b,c, 1996), De Vylder & Goovaerts (1982, 1983a,b), Goovaerts, Haezendonck & De Vylder (1982) and Goovaerts, De Vylder & Haezendonck (1984). They considered the problem within the framework of finite-dimensional convex analysis. A main topic in convex analysis, and in convex optimization in particular, is to find the extremes of a (quasi-)convex function on a finite-dimensional convex body. The interested reader is referred to Ekeland & Temam (1976), Ioffe & Tikhomirov (1979), Tikhomirov (1996) and Borwein & Lewis (2000) for further details on finite-dimensional convex analysis.

A different approach is taken for the *reduced problem of moments*, which was studied already by Markov (1884) and Riesz (1911) in the late nineteenth and early twentieth century. Here, n moments $\mu_1, \mu_2, \ldots, \mu_n$ are given, and one tries to find extremal values of the expected value of a certain function of the risk having these moments. An example is the stop-loss premium $\mathbb{E}[(X-t)_+]$ for a certain fixed t, or simply the tail probability $\mathbb{P}[X > t]$. The existence of a feasible distribution satisfying these moments constraints can be expressed by means of a quadratic form that has to be non-negative; see, e.g., Theorem 2.1.3 in Kaas (1987). In case the moment problem allows a solution, it can be calculated by the construction of polynomials that are upper (or lower) bounds for the function involved on the interval considered. Choosing the optimal polynomials boils down to finding polynomials that are tangent to the function considered. The points of intersection of the polynomials and the function considered are just the support of the extremal distribution. For different intervals in which the support must lie, e.g., $[0, +\infty)$, different solutions are obtained.[2] The integral constraints here are restricted to be moment constraints of type $\mathbb{E}[X^k] = \mu_k$. An early reference is Shohat & Tamarkin (1943); see also Isii (1960, 1963) and Kemperman (1968). Generalizations of this scheme are the Tchebycheff systems; see Karlin & Studden (1966). The general problem studied in this paper is also related to mass transportation problems (MTP); see Rachev & Rüschendorf (1998) for a detailed account, and also Sections 5 and 6 below.

This paper revisits and extends some of the work on best-possible upper bounds for integrals with respect to measures, done by Etienne De Vijlder and his co-authors. The extension is twofold: (i) we study some topical univariate risk measurement problems; and (ii) we demonstrate that the general theory considered by Etienne De Vijlder and his

[1]Etienne's full name was "De Vijlder, F.E.C. (Florian Etienne Charles)", but for references to his papers we use his *nom de plume* "De Vylder".

[2]There are three named "classical" moment problems: the *Hamburger* moment problem in which the support of the distribution is allowed to be the whole real line; the *Stieltjes* moment problem, for $[0, +\infty)$; and the *Hausdorff* moment problem for a bounded interval, which without loss of generality may be taken as $[0, 1]$.

co-authors to a large extent allows application to worst case measurement of multivariate risk, while only applications to problems of univariate risk have been considered hitherto.

Other contributions to the problem of worst case measurement of univariate risk with applications to insurance and finance in mind include among others, Taylor (1977), Goovaerts & Kaas (1985), Brockett & Cox (1985) and Denuit, De Vylder & Lefèvre (1999).

The problem of finding best-possible upper bounds on measures of multivariate risk when the marginal distributions are known, has a rich history in probability theory, where it typically appears under the name *Fréchet problem*. The problem of bounding the distribution function of a sum of random variables with given marginal distributions can be traced back to A.N. Kolmogorov. It was solved by Makarov (1981) in a two-dimensional setting, and, using different routes, by Rüschendorf (1982, dual approach) and Frank, Nelsen & Schweizer (1987, copula-based approach). Recent contributions to the problem of multivariate risk measurement under incomplete information with given marginals include Denuit, Genest & Marceau (1999), Kaas, Dhaene & Goovaerts (2000), Dhaene *et al.* (2002), Rüschendorf (2004), Denuit *et al.* (2005), Embrechts, Höing & Puccetti (2005), Embrechts & Puccetti (2006), Kaas, Laeven & Nelsen (2009) and Laeven (2009).

The rest of this paper is organized as follows: Section 2 presents the general problem, and recalls and discusses some main results. In Section 3, we introduce the *special dual problem* to which the general problem can often be reduced. In Section 4, we consider a discretization method to solve the special dual problem numerically. Section 5 discusses some strengths and shortcomings of the dual theory presented. In Section 6, we illustrate the dual approach with some univariate as well as multivariate examples. Finally, Section 7 contains some concluding remarks.

2 The general problem

We consider a measurable space $(\Omega, \mathcal{A})$ and $\mathcal{A}$-measurable functions $f, g_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$. Furthermore, we consider a cone $\mathcal{M}$ of countably additive measures $\mu : \mathcal{A} \to \mathbb{R} \cup \{-\infty, +\infty\}$. The cone $\mathcal{M}$ need not be pointed, i.e., it does not necessarily contain the zero measure. We note that a cone need not be a convex set; in many applications, however, the cone $\mathcal{M}$ will contain non-negative measures only, in which case $\mathcal{M}$ is a true convex set. All integrals in this section are over Ω.

We consider the following problem:

$$v(\mathbf{c}) = \sup_{\mu \in \mathcal{M}} \left(\int f d\mu \quad | \quad \int g_i d\mu = c_i, \quad i = 1, \ldots, n \right), \tag{1}$$

with the real-valued vector $\mathbf{c} = (c_1, \dots, c_n)$ and the functions $f, g_i, i = 1, \dots, n$, fixed and given. We assume that $\int f d\mu$ and $\int g_i d\mu$, $i = 1, \dots, n$, exist for all $\mu \in \mathcal{M}$. Henceforth, we refer to $v(\mathbf{c})$ as the *primal problem*. Furthermore, we call the constraints under the sup-symbol the *conical constraints* and the other constraints the *integral constraints*. A measure μ that satisfies all constraints is called *feasible*. Notice that the set of all feasible measures is typically not a cone. The conical property of $\mathcal{M}$ is mainly imposed to allow a transformation of the general problem to a better tractable problem, which will be introduced below. In fact, the conical assumption can easily be nullified by imposing an integral constraint. A feasible μ that attains the supremum of the problem $v(\mathbf{c})$ is a *solution*. Notice that there exists a collection of problems $v(\mathbf{c})$, one for each $\mathbf{c} \in \mathbb{R}^n$. By problem v we denote this collection of problems. Note the generality of the problem v.

The primal maximization problem can be shown to be associated with a *dual* minimization problem of linear programming type. In many cases, solving the dual problem amounts to determining the convex hull of a set in $\mathbb{R}^n$, specified by the constraints of the primal problem.

Below we summarize some important results on problem (1); for the proofs of the lemmas in this and the following section, we refer to De Vylder (1982).

Lemma 2.1 *The primal problem v is strictly positively homogeneous, i.e., $v(\alpha\mathbf{c}) = \alpha v(\mathbf{c})$ whenever $\alpha > 0$.* □

Lemma 2.2 *The primal problem v is concave if $\mathcal{M}$ is a convex cone, i.e., $v(\alpha\mathbf{c} + (1 - \alpha)\mathbf{c}') \geq \alpha v(\mathbf{c}) + (1 - \alpha)v(\mathbf{c}')$ with $0 \leq \alpha \leq 1$ whenever the cone $\mathcal{M}$ is convex.* □

In the following, we denote by $\mathcal{F}$ the subspace of all linear combinations of $f, g_1, \dots, g_n$. Furthermore, we denote by $\mathcal{F}_{\mathcal{M}}$ the subset of $\mathcal{F}$ defined by

$$\mathcal{F}_{\mathcal{M}} = \left\{ h \in \mathcal{F} \quad | \quad \int h d\mu \geq 0, \quad \mu \in \mathcal{M} \right\}. \tag{2}$$

Then, we consider the problem $\tilde{v}$ defined by[3]

$$\tilde{v}(\mathbf{c}) = \inf_{\mathbf{d} \in \mathbb{R}^n} \left(\sum_{i=1}^n d_i c_i \quad | \quad \sum_{i=1}^n d_i g_i - f \in \mathcal{F}_{\mathcal{M}} \right), \qquad \mathbf{c} \in \mathbb{R}^n, \tag{3}$$

with $\inf(\emptyset) = +\infty$ by convention. Similar to the terminology introduced above, a vector $\mathbf{d}$ that satisfies the constraints of problem $\tilde{v}(\mathbf{c})$ is called *feasible*, and a feasible vector $\mathbf{d}$

[3]The problem $\tilde{v}$ is the convex conjugate of the convex conjugate of v; it is also referred to as the Fenchel-Legendre transform of the Fenchel-Legendre transform of v, or the biconjugate or bipolar function of v.

that attains the infimum of the problem $\tilde{v}(\mathbf{c})$ is called a *solution*. We state the following lemmas:

Lemma 2.3 *The problem $\tilde{v}$ is strictly positively homogeneous. The problem $\tilde{v}$ is concave (regardless of whether $\mathcal{M}$ is convex).* □

Lemma 2.4 *We have that*

$$v(\mathbf{c}) \leq \tilde{v}(\mathbf{c}), \qquad \mathbf{c} \in \mathbb{R}^n. \tag{4}$$

□

Lemma 2.4 establishes an important relation between the values of the problems v and $\tilde{v}$. Under specific conditions, which will be stated below, inequality (4) even turns out to be an equality. However, in the following, we will need a subtle definition of "equality". For that purpose, we first introduce some preliminaries.

We say that the concave problem v is *proper* if $v < +\infty$ and $v \neq -\infty$, i.e., if v never takes the value $+\infty$ and there exists a vector $\mathbf{c} \in \mathbb{R}^n$ such that $v(\mathbf{c}) \neq -\infty$.

Lemma 2.5 *Let v be concave on $\mathbb{R}^n$. Then v is proper if and only if $\tilde{v}(\mathbf{c})$ is finite for some $\mathbf{c} \in \mathbb{R}^n$.* □

Henceforth, we denote by $\mathrm{Dom}[v]$ the *effective domain* of the problem v defined by

$$\mathrm{Dom}[v] = \{\mathbf{c} \in \mathbb{R}^n \quad | \quad v(\mathbf{c}) > -\infty\}. \tag{5}$$

Notice that if v is concave, then $\mathrm{Dom}[v]$ is a convex set. Let v be concave. Then v and $\tilde{v}$ (which is concave as well) are said to be equal *almost everywhere* (a.e) if $\mathrm{Clo}\,[\mathrm{Dom}\,[v]] = \mathrm{Clo}\,[\mathrm{Dom}\,[\tilde{v}]]$ and $v(\mathbf{c}) = \tilde{v}(\mathbf{c})$ whenever $\mathbf{c} \in \mathrm{Int}\,[\mathrm{Dom}\,[v]]$. Here, for any set $A \in \mathbb{R}^n$, we denote by $\mathrm{Clo}[A]$ and $\mathrm{Int}[A]$ its *closure* and *interior*, respectively. We write $v = \tilde{v}$, a.e. The terminology "almost everywhere" is closely related to the same terminology used in measure theory, since the boundary of any convex set in $\mathbb{R}^n$ can be proven to have Lebesgue measure zero. From a topologist's perspective, the boundary of a convex set in a linear topological space L is nowhere dense in L (Fan, 1959). Then we state the following lemma:

Lemma 2.6 *If v is concave and proper (in particular, if $\mathcal{M}$ is convex and $\tilde{v}(\mathbf{c})$ is finite for some $\mathbf{c} \in \mathbb{R}^n$), then $v = \tilde{v}$, a.e.* □

The following lemma holds as a result of Lemma 2.4:

Lemma 2.7 *Let μ be a feasible measure of problem $v(\mathbf{c})$ and let $\mathbf{d}$ be a feasible vector of problem $\tilde{v}(\mathbf{c})$. Then the following statements are equivalent:*

(i) μ is a solution to problem $v(\mathbf{c})$, $\mathbf{d}$ is a solution to problem $\tilde{v}(\mathbf{c})$ and $v(\mathbf{c}) = \tilde{v}(\mathbf{c})$;

(ii) $\int f d\mu = \sum_{i=1}^{n} d_i c_i = \int \left(\sum_{i=1}^{n} d_i g_i\right) d\mu$. □

Clearly, by the result of Lemma 2.6, if $\mathbf{c} \in \mathrm{Int}\,[\mathrm{Dom}\,[v]]$ and v is concave and proper, then the condition that $v(\mathbf{c}) = \tilde{v}(\mathbf{c})$ in (i) of Lemma 2.7 can be left out. For obvious reasons, the problem $\tilde{v}$ will be referred to as the *dual* problem of the primal problem v.

Now we will consider the general problem in a more restrictive setting, imposing specific assumptions on the cone of measures under consideration. We assume that $\mathcal{A}$ contains all singleton sets $\{\omega\}, \omega \in \Omega$. Recall that a measure is a set function. The measure μ is said to be (k-)*atomic* if for some $k = 0, 1, 2, \ldots$ there exist a sequence $\omega_1, \ldots, \omega_k \in \Omega$ and a sequence $p_1, \ldots, p_k \in \mathbb{R}$ such that

$$\mu(A) = \sum_{\omega_i \in A} p_i, \qquad A \in \mathcal{A}, \tag{6}$$

where the summation is over the subscripts $i = 1, \ldots, k$. Since some of the p_i may be zero, a k-atomic measure is also a $(k+1)$-atomic measure (and hence a $(k+n)$-atomic measure with $n \in \mathbb{N}$). An atom of the measure μ is a singleton $\omega \in \Omega$ for which $\mu(\{\omega\}) \neq 0$. A k-atomic measure has at most k atoms.

In the remainder of this section, we assume that $\mathcal{M}$ is a convex cone of non-negative measures containing all non-negative atomic measures. We denote by $\mathcal{M}_n$ the subcone of $\mathcal{M}$ consisting of all non-negative n-atomic measures. Notice that the subcone $\mathcal{M}_n$ will not in general be convex. Furthermore, we denote by v_n the primal problem corresponding to the cone $\mathcal{M}_n$. Under these additional conditions on $\mathcal{M}$, we have the following results:

Lemma 2.8 *The primal problem v_n is strictly positively homogeneous. If v is proper (in addition to being concave which is implied by the assumption that $\mathcal{M}$ is convex), then v_n is concave and $v_n < +\infty$.* □

Lemma 2.9 *We have that*

$$\tilde{v}(\mathbf{c}) = \tilde{v}_n(\mathbf{c}), \qquad \mathbf{c} \in \mathbb{R}^n. \tag{7}$$

□

Lemma 2.10 *If v is proper, then*

$$v = \tilde{v} = \tilde{v}_n = v_n, \qquad \text{a.e.} \tag{8}$$

$\square$

Lemma 2.10 is connected to the early work of Rogosinsky (1958).

Lemma 2.11 *Let v be proper and let $\mathbf{c}$ be fixed in the interior of the domain of v. Suppose that μ is a solution to the problem $v(\mathbf{c})$ and that $\mathbf{d}$ is a solution to the problem $\tilde{v}(\mathbf{c})$. Then each atom ω of the measure μ is a root of the atoms equation*

$$\sum_{i=1}^{n} d_i g_i(\omega) = f(\omega). \tag{9}$$

$\square$

Often, the atoms equation (9) enables us to find the atoms of an n-atomic solution to the primal problem $v(\mathbf{c})$. The masses corresponding to the atoms can then be found by solving the integral constraints. Notice that in order to solve the atoms equation, the vector $\mathbf{d}$ that solves the dual problem $\tilde{v}(\mathbf{c})$ must be known. In the following section, we present a method to solve an important class of dual problems.

Under the conditions stated in the following lemma, there exists an n-atomic solution to the primal problem $v(\mathbf{c})$:

Lemma 2.12 *The problem $v(\mathbf{c})$ has an n-atomic solution if the following conditions are satisfied:*

(i) *Ω is a compact subset of some $\mathbb{R}^m$ and the usual topology is used on Ω;*

(ii) *$\mathcal{A}$ is the σ-algebra generated by the collection of open subsets of Ω, i.e., $\mathcal{A}$ is the Borel σ-algebra on Ω;*

(iii) *the functions f, g_i, $i = 1, \ldots, n$, are continuous;*

(iv) *the concave problem v is proper;*

(v) *c is in the interior of the domain of v;*

(vi) *$\mu(\Omega) = 1$ is one of the constraints of the problem $v(\mathbf{c})$.* $\square$

Remark 2.1 *Consider the minimization problem*

$$w(\mathbf{c}) = \inf_{\mu \in \mathcal{M}} \left(\int f d\mu \quad \Big| \quad \int g_i d\mu = c_i, \quad i = 1, \ldots, n \right), \qquad \mathbf{c} \in \mathbb{R}^n. \tag{10}$$

Notice that a minimization problem can as well be expressed as a maximization problem by alternating the sign. Hence, the results derived in this section can easily be adapted to the case of problem (10).

3 The special dual problem

In many cases, the dual version of the general problem is of a particular common form. This particular form, which will be introduced below, will be referred to as the *special dual problem.*

Let Θ be a set and let $f_\Theta, g_{i,\Theta} : \Theta \to \mathbb{R}$, $i = 1, \ldots, n-1$. We introduce the *special dual problem* $\tilde{v}_\Theta$ defined by

$$\tilde{v}_\Theta(\mathbf{c}_{-n}) = \inf_{\mathbf{d} \in \mathbb{R}^n} \left(\sum_{i=1}^{n-1} d_i c_i + d_n \quad \Big| \quad \sum_{i=1}^{n-1} d_i g_{i,\Theta}(\theta) + d_n \geq f_\Theta(\theta), \quad \theta \in \Theta \right), \tag{11}$$

with $\mathbf{c}_{-n} = (c_1, \ldots, c_{n-1}) \in \mathbb{R}^{n-1}$ and the functions $f_\Theta, g_{i,\Theta}, i = 1, \ldots, n-1$, fixed and given. We refer to Section 6 for examples of primal problems of which the dual version can be cast into the form (11), with $\Theta = \Omega$, $f_\Theta = f$ and $g_{i,\Theta} = g_i$, $i = 1, \ldots, n-1$, and with $\mu(\Omega) = 1$ among the integral constraints of the primal problem. Notice that the special dual problem is a minimization problem of linear programming type, but possibly with an infinite number of linear inequality constraints.

Problem (11) also appeared in Charnes, Cooper & Kortanek (1962) as the primal problem of a dual pair of semi-infinite problems; see also Hettich & Kortanek (1993). The concept of considering all program values of a given linear semi-infinite programming problem has also been studied within the context of uniform linear programming duality by Duffin, Jeroslaw & Karlovitz (1983).

In problems of univariate risk measurement, the set $\mathcal{M}$ of measures will often include all 1-atomic measures with mass 1 at a particular $\omega \in \Omega$. If furthermore the constraint $\mu(\Omega) = 1$ is among the integral constraints of the primal problem, then the general dual problem (3) always reduces to the special dual problem (11).

To analyze the special dual problem in further detail, we first introduce several concepts. We define the set $E_\Theta \subseteq \mathbb{R}^n$ by

$$E_\Theta = \{(g_{1,\Theta}(\theta), \ldots, g_{n-1,\Theta}(\theta), f_\Theta(\theta)) \quad | \quad \theta \in \Theta\}. \tag{12}$$

Furthermore, for any set E, we define by $\mathrm{Co}[E]$ its *convex hull*, being the smallest convex set containing E, that is, the intersection of all convex sets containing E. Next, for a given vector $\mathbf{d} \in \mathbb{R}^n$, we introduce the *affine function* $l_{\mathbf{d}}$ on $\mathbb{R}^{n-1}$, defined by

$$l_{\mathbf{d}}(\mathbf{x}_{-n}) = \sum_{i=1}^{n-1} d_i x_i + d_n, \qquad \mathbf{x}_{-n} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}. \tag{13}$$

Notice that the special dual problem $\tilde{v}_\Theta$ can be expressed as follows:

$$\tilde{v}_\Theta(\mathbf{c}_{-n}) = \inf_{\mathbf{d}\in\mathbb{R}^n} \left(l_{\mathbf{d}}(\mathbf{c}_{-n}) \quad \middle| \quad l_{\mathbf{d}}(g_{1,\Theta}(\theta), \ldots, g_{n-1,\Theta}(\theta)) \geq f_\Theta(\theta), \quad \theta \in \Theta \right), \qquad \mathbf{c}_{-n} \in \mathbb{R}^{n-1}.$$

An *affine majorant* of a given set $E \subseteq \mathbb{R}^n$, is an affine function $l_{\mathbf{d}}$ on $\mathbb{R}^{n-1}$ such that $l_{\mathbf{d}}(\mathbf{x}_{-n}) \geq x_n$ for all $(x_1, \ldots, x_n) \in E$.

For any given $\mathbf{c}_{-n} \in \mathbb{R}^{n-1}$, we then consider the *upper envelope problem*

$$e_\Theta(\mathbf{c}_{-n}) = \sup_{c\in\mathbb{R}} \left(c \quad | \quad (\mathbf{c}_{-n}, c) \in \mathrm{Co}\left[E_\Theta\right] \right). \tag{14}$$

We state the following lemma:

Lemma 3.1 *The problems $\tilde{v}_\Theta$ and e_Θ are concave. If one of them is proper, the other is proper as well, and*

$$\tilde{v}_\Theta = e_\Theta, \quad \text{a.e.} \tag{15}$$

□

Hence, for values $\mathbf{c}_{-n}$ in the interior of the domain of $\tilde{v}_\Theta$, solving a proper special dual problem amounts to determining the convex hull of the set E_Θ. Furthermore, we have the following result on the solution to the special dual problem:

Lemma 3.2 *If $\mathbf{c}_{-n}$ is in the interior of the domain of $\tilde{v}_\Theta$ and $\tilde{v}_\Theta$ is proper, then $\mathbf{d}$ is a solution to problem $\tilde{v}_\Theta(\mathbf{c}_{-n})$ if and only if $l_{\mathbf{d}}$ is an affine majorant of e_Θ, being exact at $\mathbf{c}_{-n}$.* □

It is an immediate consequence of Lemma 3.2 that if $\tilde{v}_\Theta$ is proper, the problem $\tilde{v}_\Theta(\mathbf{c}_{-n})$ has at least one solution for each $\mathbf{c}_{-n}$ in the interior of the domain of $\tilde{v}_\Theta$.

4 Discretization of the special dual problem

The tractability of the special dual problem decreases when the number of integral constraints increases. In particular, if the number of integral constraints (including the constraint that $\int d\mu = 1$) gets larger than three, the visual geometry of the dual approach disappears and the determination of the convex hull of E_Θ becomes more complex. In this section, we show how, for any n, the special dual problem can be solved numerically. Consider again the special dual problem $\tilde{v}_\Theta(\mathbf{c}_{-n})$ defined in (11). Below, we summarize some important discretization results for this problem; the proofs of the lemmas can be found in Goovaerts, Haezendonck & De Vylder (1982).

Lemma 4.1 *Let $\Theta_1, \Theta_2, \ldots$ be an increasing sequence of non-empty subsets of Θ, with*

$$\Theta_{+\infty} := \bigcup_{i=1}^{+\infty} \Theta_i. \tag{16}$$

Then the problems $\tilde{v}_\Theta$,$\tilde{v}_{\Theta_{+\infty}}$,$\tilde{v}_{\Theta_1}$,$\tilde{v}_{\Theta_2}$,... are concave and $\tilde{v}_{\Theta_1} \le \tilde{v}_{\Theta_i} \le \tilde{v}_{\Theta_{+\infty}} \le \tilde{v}_\Theta$, $i = 1, 2, \ldots$ Furthermore, the limit

$$\lim_{i \to +\infty} \tilde{v}_{\Theta_i} := \tilde{v}_\Theta \nearrow \tag{17}$$

is concave. □

Lemma 4.2 *If $E_{\Theta_{+\infty}}$ is dense in E_Θ and $\tilde{v}_\Theta$ is proper, then $\tilde{v}_\Theta \nearrow = \tilde{v}_\Theta$, a.e. If furthermore $\mathbf{c}_{-n}$ is in the interior of the domain of $\tilde{v}_\Theta$, $\mathbf{d}_i \in \mathbb{R}^n$ is a solution to problem $\tilde{v}_{\Theta_i}$ and $\mathbf{d}_i$ converges to $\mathbf{d}$ when i tends to $+\infty$, then $\mathbf{d}$ is a solution to the problem $\tilde{v}_\Theta(\mathbf{c}_{-n})$.* □

Hence, by the result of Lemma 4.2, for i large enough, the problems $\tilde{v}_{\Theta_i}$ are considered to be approximations to the problem $\tilde{v}_\Theta$. Notice that if the Θ_i's are finite sets, the problems $\tilde{v}_{\Theta_i}$ are ordinary discrete linear programming problems that can be solved by one of the well-known methods for solving linear programming problems such as Dantzig's simplex algorithm; see e.g., Hillier & Lieberman (2001).

The convergence theorem employed in Hettich (1986) states that if the level sets of the special dual problem (11) are compact, then by taking a sufficiently fine subset of Θ, each solution of the discrete problem can be driven arbitrarily close to one of the original problem.

4.1 A numerical illustration using R

To illustrate the numerical tractability of the special dual problem, also in a setting with more than three constraints in which solving the upper envelope problem becomes more complex, we consider the following problem:

$$\tilde{v}_{[0,1000]}(10, 2100, 0.05) = \inf_{\mathbf{d} \in \mathbb{R}^4} \big(10d_1 + 2100d_2 + 0.05d_3 + d_4 \mid d_1 x + d_2 x^2 + d_3 1_{\{x \ge t\}} + d_4 \ge \max(x - t, 0),\ 0 \le x \le 1000\big);$$

cf. Goovaerts, Haezendonck & De Vylder (1982), Problem P^1_{123}. Here, as usual, we denote by 1_A the *indicator function* of event A. We solve the problem numerically using R (2011, version 2.13.0), taking $t = 100$. It requires the R packages linprog (Henningsen, 2010) and lpSolve (Berkelaar *et al.*, 2011):

```
library(linprog)
t<-100
val<-function(x) ## to minimize c'd s.t. A d >= b
{c <- c(10,2100,0.05,1); b <- max(x-t,0); A <- t(c(x,x^2,x>t,1))
solveLP(c, b, A, maximum=FALSE, ">=")$opt}
optimize(val, lower=0, upper=1000, maximum=TRUE)
```

The obtained optimal function value (5.238095) is to be compared with the fourth line in Table 1 of Goovaerts, Haezendonck & De Vylder (1982), where regardless of the mesh of the discretization the value 5.0 is reported. It makes explicit the advancement of (the accessibility of) computational methods over the past 30 years.

5 Discussion

Some remarks:

- In general, the dual approach is a powerful tool for solving problems of worst case risk measurement. Many worst case risk measurement problems can be cast into the form of the special dual problem.

- However, notice that if the dual problem is not of the form of the *special* dual problem, the dual formulation does not help in general to solve the primal problem. An example of a primal problem of which the dual problem is not of the form of the special dual problem is the following (cf. the example in Section 6.8 below which does allow a special dual problem representation):

  $\Omega = [0,1]^2$, $\mathcal{A}$ is the Borel σ-algebra on Ω, $\mathcal{M}$ consists of elements μ that are associated with a copula through $\mu([0,x] \times [0,y]) = C(x,y)$ with C a copula, $f(x,y) = 1_{\{x+y\geq t\}}$ for a given t, $n = 2$, $g_1(x,y) = xy$, $\frac{1}{6} \leq c_1 \leq \frac{1}{3}$, $g_2(x,y) = 1$ and $c_2 = 1$.

  The interested reader is referred to Kaas, Laeven & Nelsen (2009) for a solution to this primal problem.

- If the number of integral constraints (including the constraint that $\int d\mu = 1$) becomes larger than three, the determination of the convex hull of E_Θ becomes more complex and we have to resort to numerical methods to solve the special dual problem.

- De Vylder (1983a) also generalized the results, aiming to find the maximal possible variance of the stop-loss payment $(X - t)_+$ given the mean and variance of X. He provided the general theoretical framework for maximization of functionals of the distribution of the risk, for instance the mean square minus the squared mean of a function of the risk; see also De Vylder & Goovaerts (1983a).

- In a multivariate setting with fixed and given marginal distributions, the dual problem features an objective function that is maximized over a class of measurable functions, hence is infinite-dimensional, next to infinite-dimensional constraints. Such problems are connected to (may take the form of) Monge-Kontorovich mass transportation problems, studied in detail in Rachev & Rüschendorf (1998). A rich duality theory is available for these problems.

6 Examples

In this section, we present several examples of worst case risk measurement.

6.1 Example 1: Maximal stop-loss premiums

As a simple illustration of the results derived in Sections 2 and 3, we consider the problem of finding the maximal stop-loss premiums for a random variable with given mean.

Let $\Omega = [a, b]$, with $a < b$, and let $\mathcal{A}$ be the Borel σ-algebra on Ω. We consider the convex cone $\mathcal{M}$ of countably additive (or σ-additive), non-negative and finite measures on $\mathcal{A}$, including the atomic ones. With a given measure $\mu \in \mathcal{M}$, we identify the function $F : [a, b] \to \mathbb{R}_+$ as follows:

$$F(x) = \mu([a, x]), \qquad x \in [a, b]. \tag{18}$$

Notice that F is non-decreasing, right-continuous and bounded. Notice furthermore, that for any continuous function $f : [a, b] \to \mathbb{R}$

$$\int_{[a,b]} f d\mu = \int_{a-}^{b} f(x) dF(x), \tag{19}$$

where the integral on the right-hand side of (19) is a Riemann-Stieltjes integral. Let $\mathcal{G}$ be the set of functions $F : [a, b] \to \mathbb{R}_+$, which correspond one-to-one to an element in the set $\mathcal{M}$. Then, for given numbers c_1, t, with $a < c_1, t < b$, we consider the following primal problem:

$$v(c_1, 1) = \sup_{F \in \mathcal{G}} \left(\int_{a-}^{b} (x-t)_+ dF(x) \quad | \quad \int_{a-}^{b} x dF(x) = c_1, \int_{a-}^{b} dF(x) = 1 \right), \tag{20}$$

with $(x-t)_+ := \max(x-t, 0)$. In case F is a distribution function, the expression $\int_{a-}^{b} (x-t)_+ dF(x)$ is known as the *stop-loss premium* of F with a *retention level* equal to t. This problem was considered first by Gagliardi & Straub (1974). It is not difficult to verify that the dual problem $\tilde{v}(c_1, 1)$ is given by

$$\tilde{v}(c_1, 1) = \inf_{\mathbf{d} \in \mathbb{R}^2} \left(d_1 c_1 + d_2 \quad | \quad d_1 x + d_2 \geq (x-t)_+, \quad a \leq x \leq b \right). \tag{21}$$

Notice that the problem in (21) has the particular form of the special dual problem defined in (11), with $n = 2$, $\Theta = \Omega = [a, b]$, $f_\Theta(x) = f(x) = (x-t)_+$ and $g_{\Theta,1}(x) = g_1(x) = x$.

To determine the value $v(c_1, 1)$, we follow the approach of Section 3. The set $E_{[a,b]}$ is given by

$$E_{[a,b]} = \{(x, (x-t)_+) \quad | \quad a \leq x \leq b\}, \tag{22}$$

and corresponds to the bold line in Figure 1. The convex hull $\mathrm{Co}\left[E_{[a,b]}\right]$ corresponds to the triangle in Figure 1. Furthermore, $e_{[a,b]}(\cdot)$ corresponds to the dotted line in Figure 1.

Since $e_{[a,b]}$ is proper, we have by Lemma 3.1 that

$$\tilde{v}_{[a,b]}(c_1) = e_{[a,b]}(c_1) = (c_1 - a)\frac{b-t}{b-a}, \tag{23}$$

and by Lemma 2.6 that

$$\tilde{v}_{[a,b]}(c_1) = v(c_1, 1). \tag{24}$$

We have thus found the value of problem $v(c_1, 1)$.

To find the function F that solves the primal problem (20), we proceed as follows: Lemma 3.2 implies that if $l_{\mathbf{d}}$ is an affine majorant of $e_{[a,b]}$ with $l_{\mathbf{d}}(c_1) = e_{[a,b]}(c_1)$, then $\mathbf{d}$ is a solution to the problem $\tilde{v}_{[a,b]}(c_1)$. Hence, we solve

$$d_1 c_1 + d_2 = (c_1 - a)\frac{b-t}{b-a}, \tag{25}$$

for d_1, d_2. This gives the unique solution

$$d_1 = \frac{b-t}{b-a}, \qquad d_2 = -a\frac{b-t}{b-a}. \tag{26}$$

Notice that for these particular values of d_1, d_2, the function $l_{\mathbf{d}}$ is an affine majorant of $e_{[a,b]}$.

The existence of a 2-atomic function F that solves problem (20) is guaranteed by Lemma 2.12. The atoms of the function F (i.e., the set $\{x \in [a,b] \quad | \quad dF(x) > 0\}$) can be derived by solving the atoms equation

$$\frac{b-t}{b-a}x - a\frac{b-t}{b-a} = (x-t)_+ \tag{27}$$

for x; see also (9). The roots are a and b. Finally, to find the masses corresponding to these atoms, we solve the integral constraints

$$p_a + p_b = 1, \qquad p_a a + p_b b = c_1, \tag{28}$$

where $p_a = F(a)$ and $p_b = F(b) - F(a)$. This gives

$$p_a = \frac{b-c_1}{b-a}, \qquad p_b = \frac{c_1-a}{b-a}. \tag{29}$$

Notice that for the derived values of d_1, d_2 and p_a, p_b, the equality (ii) of Lemma 2.7 holds. This confirms that we have indeed found a solution.

The interested reader is referred to Section 10.5 of Kaas *et al.* (2001) for an alternative proof that a solution to (20) is the unique 2-atomic measure defined in (29).

6.2 Example 2: Maximization over a cone of m-unimodal measures

In this subsection, we consider problem (1) under specific restrictions on the cone $\mathcal{M}$ of measures under consideration. As before, we let $\Omega = [a,b]$ and let $\mathcal{A}$ be the Borel σ-algebra on Ω. Throughout this section, we denote by $F : [a,b] \to \mathbb{R}_+$ a non-decreasing and right-continuous function.

We say that F is *m-unimodal* if it is convex on $[a,m)$ and concave on $[m,b]$; see Feller (1971) for further details on unimodality. On the measurable space $(\Omega, \mathcal{A})$, we then restrict ourselves to measures μ that are associated with an m-unimodal function F through equality (18). We will refer to such a μ as an *m-unimodal measure*.

It is straightforward to verify that any m-unimodal function F can be expressed as the sum of an absolutely continuous, m-unimodal function F_1 and a one-step function F_2 with non-negative step at m. We say that F is *absolutely continuous* if

$$F(x) = \int_a^x \varphi(s)ds, \qquad x \in [a,b], \tag{30}$$

for some $\mathcal{A}$-measurable function $\varphi : [a,b] \to \mathbb{R}_+$. Notice that an absolutely continuous function F is m-unimodal if and only if φ is non-decreasing on $[a,m)$ and non-increasing on $(m,b]$.

In the following, we denote by 1_ξ, with $\xi \in [a,b]$, the special indicator function defined by

$$1_\xi(x) = \begin{cases} 1_{\{x \in [\xi,m]\}}, & \xi \le m; \\ 1_{\{x \in [m,\xi]\}}, & \xi \ge m. \end{cases} \tag{31}$$

Suppose that

$$\varphi(x) = \sum_{j=1}^{k} p_j 1_{\xi_j}(x), \tag{32}$$

with $p_j \ge 0$, $\xi_j \in [a,b]$, $j = 1, \ldots, k$. Then the corresponding absolutely continuous function F defined through (30) is clearly m-unimodal, and is furthermore called *k-rectangular*. A k-rectangular, m-unimodal function F is either absolutely continuous or the sum of an absolutely continuous, $(k-1)$-rectangular, m-unimodal function F_1 and a one-step function F_2 with positive step at m. Notice that any k-rectangular function F is also $(k+1)$-rectangular (and hence $(k+n)$-rectangular, with $n \in \mathbb{N}$).

In the following, we will distinguish four cones:

1. $\mathcal{G}_{u(m)} := \{F \quad | \quad F \text{ is } m\text{-unimodal}\}$;

2. $\mathcal{G}_{ac,u(m)} := \{F \quad | \quad F \text{ is absolutely continuous, } m\text{-unimodal}\}$;

3. $\mathcal{G}_{r(n),u(m)} := \{F \quad | \quad F \text{ is } n\text{-rectangular, } m\text{-unimodal}\}$;

4. $\mathcal{G}_{ac,r(n),u(m)} := \{F \quad | \quad F \text{ is absolutely continuous, } n\text{-rectangular, } m\text{-unimodal}\}$.

Here, as before, n denotes the number of integral constraints in the primal problem. Notice that $\mathcal{G}_{u(m)}$ and $\mathcal{G}_{ac,u(m)}$ are convex cones. To derive the dual problem corresponding to the primal problem for the four cones, we will use auxiliary functions. We introduce the auxiliary functions $\tilde{f}, \tilde{g}_i : [a,b] \to \mathbb{R}$, $i = 1, \ldots, n$, defined by

$$\tilde{f}(x) = \int_a^b 1_x(y) f(y) dy, \qquad \tilde{g}_i(x) = \int_a^b 1_x(y) g_i(y) dy, \tag{33}$$

assuming that f, g_i, $i = 1, \ldots, n$, are integrable over all compact subsets of $[a,b]$. Notice that if F is absolutely continuous, k-rectangular, then

$$\int_a^b f(x) dF(x) = \sum_{j=1}^{k} p_j \tilde{f}(\xi_j). \tag{34}$$

Henceforth, we denote by $v_{u(m)}, v_{ac,u(m)}, v_{r(n),u(m)}, v_{ac,r(n),u(m)}$ the primal problems corresponding to the cones $\mathcal{G}_{u(m)}, \mathcal{G}_{ac,u(m)}, \mathcal{G}_{r(n),u(m)}, \mathcal{G}_{ac,r(n),u(m)}$, respectively. We state the following lemmas, of which the proofs can be found in De Vylder (1982):

Lemma 6.1 *If the concave problem $v_{u(m)}$ is proper, then the problems $v_{r(n),u(m)}$ and $v_{ac,r(n),u(m)}$ are concave and $v_{ac,u(m)}, v_{r(n),u(m)}, v_{ac,r(n),u(m)} < +\infty$.* $\square$

Lemma 6.2

$$\mathcal{F}_{\mathcal{G}_{u(m)}} = \mathcal{F}_{\mathcal{G}_{r(n),u(m)}} = \{h \in \mathcal{F} \quad | \quad \tilde{h} \geq 0,\ h(m) \geq 0\}, \tag{35}$$

and

$$\mathcal{F}_{\mathcal{G}_{ac,u(m)}} = \mathcal{F}_{\mathcal{G}_{ac,r(n),u(m)}} = \{h \in \mathcal{F} \quad | \quad \tilde{h} \geq 0\}, \tag{36}$$

with $\mathcal{F}$ and $\mathcal{F}_{\mathcal{G}}$ as defined just above and in (2), respectively. If $f, g_i, i = 1, \ldots, n$, are all right-continuous at m or all left-continuous at m, then

$$\mathcal{F}_{\mathcal{G}_{u(m)}} = \mathcal{F}_{\mathcal{G}_{ac,u(m)}} = \mathcal{F}_{\mathcal{G}_{r(n),u(m)}} = \mathcal{F}_{\mathcal{G}_{ac,r(n),u(m)}}. \tag{37}$$

$\square$

Hence, by the virtue of Lemma 6.2, we have that the dual problems for the four cones under consideration, are given by

$$\tilde{v}_{u(m)}(\mathbf{c}) = \tilde{v}_{r(n),u(m)}(\mathbf{c}) = \inf_{\mathbf{d}\in\mathbb{R}^n} \left(\sum_{i=1}^n d_i c_i \quad | \quad \sum_{i=1}^n d_i \tilde{g}_i \geq \tilde{f},\ \sum_{i=1}^n d_i g_i(m) \geq f(m) \right), \tag{38}$$

and

$$\tilde{v}_{ac,u(m)}(\mathbf{c}) = \tilde{v}_{ac,r(n),u(m)}(\mathbf{c}) = \inf_{\mathbf{d}\in\mathbb{R}^n} \left(\sum_{i=1}^n d_i c_i \quad | \quad \sum_{i=1}^n d_i \tilde{g}_i \geq \tilde{f} \right). \tag{39}$$

If furthermore $f, g_i, i = 1, \ldots, n$, are all right-continuous at m or all left-continuous at m, then

$$\tilde{v}_{u(m)} = \tilde{v}_{ac,u(m)} = \tilde{v}_{r(n),u(m)} = \tilde{v}_{ac,r(n),u(m)}. \tag{40}$$

We state the following lemmas, which are closely related to the Lemmas 2.10, 2.11 and 2.12, respectively.

Lemma 6.3 *If the concave function $v_{u(m)}$ is proper, then*

$$v_{u(m)} = \tilde{v}_{u(m)} = \tilde{v}_{r(n),u(m)} = v_{r(n),u(m)}, \qquad a.e., \tag{41}$$

and

$$v_{ac,u(m)} = \tilde{v}_{ac,u(m)} = \tilde{v}_{ac,r(n),u(m)} = v_{ac,r(n),u(m)}, \qquad a.e. \tag{42}$$

If furthermore $f, g_i, i = 1, \ldots, n$, are all right-continuous at m or all left-continuous at m, then

$$v_{u(m)}(\mathbf{c}) = v_{ac,u(m)}(\mathbf{c}) = v_{r(n),u(m)}(\mathbf{c}) = v_{ac,r(n),u(m)}(\mathbf{c}), \qquad a.e. \tag{43}$$

□

Lemma 6.4 *Let the concave problem $v_{u(m)}$ be proper and let $\mathbf{c}$ be fixed in the interior of the domain of $v_{u(m)}$. Suppose that F is an n-rectangular, m-unimodal solution to the problem $v_{u(m)}(\mathbf{c})$, and that $\mathbf{d}$ is a solution to the problem $\tilde{v}_{u(m)}(\mathbf{c})$. Then, if m is an atom of F, i.e., $dF(m) > 0$, we have that m is a root of the atoms equation*

$$\sum_{i=1}^{n} d_i g_i(x) = f(x). \tag{44}$$

Furthermore, if the continuous part of F is defined through (32), then each ξ_j with $p_j > 0$, $j = 1, \ldots, k$, is a root of the discontinuities equation

$$\sum_{i=1}^{k} d_i \tilde{g}_i(x) = f(x). \tag{45}$$

□

Lemma 6.5 *The existence of an n-rectangular, m-unimodal solution to the problem $v_{u(m)}(\mathbf{c})$ is guaranteed if the following conditions are satisfied:*

(i) *Ω is a compact interval and the usual topology is used on Ω;*

(ii) *$\mathcal{A}$ is the Borel σ-algebra on Ω;*

(iii) *the functions f, g_i, $i = 1, \ldots, n$, are continuous in the neighbourhood of m;*

(iv) *the concave problem $v_{u(m)}$ is proper;*

(v) *c is in the interior of the domain of $v_{u(m)}$;*

(vi) *$\int dF = 1$ is one of the constraints of the problem $v(\mathbf{c})$.* □

For given numbers c_1, t, with $a < c_1, t < b$, we consider the following primal problem, which is a variation on Example 1:

$$v_{u(m)}(c_1, 1) = \sup_{F \in \mathcal{G}_{u(m)}} \left(\int_{a-}^{b} (x - t)_+ dF(x) \quad \middle| \quad \int_{a-}^{b} x dF(x) = c_1, \int_{a-}^{b} dF(x) = 1 \right), \tag{46}$$

with $a < m \leq t$. The case of $t \leq m < b$ can be solved similarly. It is straightforward to verify that

$$\tilde{f}(x) = \frac{1}{2}\left((x-t)_+\right)^2, \qquad \tilde{g}_1(x) = \frac{1}{2}|x-m|(x+m), \qquad \tilde{g}_2(x) = |x-m|. \tag{47}$$

Since all f, g_1, g_2 are continuous at m, the dual problem is given by

$$\tilde{v}_{u(m)}(c_1, 1) = \inf_{\mathbf{d}\in\mathbb{R}^2} \Big(d_1 c_1 + d_2 \quad | \quad d_1 \frac{1}{2}|x-m|(x+m) + d_2|x-m| \geq \frac{1}{2}\left((x-t)_+\right)^2, \\ a \leq x \leq b \Big). \tag{48}$$

Notice that (48) can be cast into the form of the special dual problem defined in (11), with $n = 2$, $\Theta = [a, b]$, $f_\Theta(x) = \frac{\frac{1}{2}((x-t)_+)^2}{|x-m|}$ and $g_{\Theta,1}(x) = \frac{1}{2}(x+m)$, where $\frac{0}{0} = 0$ by convention. To determine the value of $\tilde{v}_{u(m)}(c_1, 1)$, we follow the approach of Section 3. The set $E_{[a,b]}$ is given by

$$E_{[a,b]} = \left\{ \left(\frac{1}{2}(x+m), \frac{\frac{1}{2}\left((x-t)_+\right)^2}{|x-m|} \right) \quad | \quad a \leq x \leq b \right\}, \tag{49}$$

and corresponds to the bold line in Figure 2. The convex hull Co $\left[E_{[a,b]}\right]$ can be determined easily. Finally, we find that $e_{[a,b]}(\cdot)$ corresponds to the dotted line in Figure 2. Since $e_{[a,b]}$ is proper, we have by Lemma 3.1 that

$$\tilde{v}_{[a,b]}(c_1) = e_{[a,b]}(c_1) = \left(c_1 - \frac{1}{2}(a+m) \right) \frac{(b-t)^2}{(b-m)(b-a)}, \tag{50}$$

and by Lemma 6.3 that

$$\tilde{v}_{[a,b]}(c_1) = v_{u(m)}(c_1, 1). \tag{51}$$

We have thus found the value of problem $v_{u(m)}(c_1, 1)$. The function F, i.e., the solution to the primal problem (46), can be found in a similar way as the solution to the problem in Example 1.

The interested reader is referred to Section 7 of Goovaerts, Kaas & Laeven (2010b) for an application of problem (46) in a solvency setting.

6.3 Example 3: Maximal tail probability

This example considers the problem of determining the maximal tail probability for a random variable with a given value of a so-called *Markovian risk measure*.

As in the previous examples, we let $\Omega = [a, b]$, $a < b$, and let $\mathcal{A}$ be the Borel σ-algebra on Ω. Furthermore, we consider again the convex cone $\mathcal{M}$ of countably additive, non-negative and finite measures on $\mathcal{A}$, including the atomic ones. In this example, we assume $a \geq 0$.

Let ϕ be a bivariate $\mathcal{A}$-measurable function satisfying

$$\phi(x,t) \geq 1_{\{x>t\}}, \qquad a < x, t < b. \tag{52}$$

We consider the following generalized version of the Markov inequality for the tail probability of a distribution function $F : [a,b] \to [0,1]$, associated with a measure $\mu \in \mathcal{M}$ through equality (18):

$$1 - F(t) \leq \int_{a-}^{b} \phi(x,t) dF(x). \tag{53}$$

Henceforth, we assume the pair (ϕ, F) to be such that the integral on the right-hand side of (53) exists, and we further assume that ϕ is continuous and non-decreasing in its first coordinate, and normalized to satisfy $\phi(a,\cdot) = 0$. Inequality (53) is a special case of the generalized Markov inequality presented in Goovaerts *et al.* (2003). In Goovaerts *et al.* (2003) it is shown how the expression on the right-hand side of (53) can be used to generate *Markovian risk measures* at probability level $1 - c_1$ (e.g., $1 - c_1 = 0.99$). Markovian risk measures are upper bounds for the quantile function (or, the Value-at-Risk) at the same probability level.

For given numbers c_1, t, with $0 < c_1$ and $a < t < b$, we consider the following primal problem:

$$v(c_1, 1) = \sup_{F \in \mathcal{G}} \left(\int_{a-}^{b} 1_{\{x \geq t\}} dF^-(x) \quad | \quad \int_{a-}^{b} \phi(x,t) dF(x) = c_1, \int_{a-}^{b} dF(x) = 1 \right), \tag{54}$$

with $\mathcal{G}$ as defined in Example 1, and F^- the left-continuous version of F. That is, we determine the maximal probability of shortfall over a given threshold t, for a random variable with a Markovian risk measure of value t at level $1 - c_1$. One easily verifies that the dual problem $\tilde{v}(c_1, 1)$ is given by

$$\tilde{v}(c_1, 1) = \inf_{\mathbf{d} \in \mathbb{R}^2} \left(d_1 c_1 + d_2 \quad | \quad d_1 \phi(x,t) + d_2 \geq 1_{\{x \geq t\}}, \quad a \leq x \leq b \right), \tag{55}$$

which has the form of the special dual problem. Let $\inf(\phi(x,t) \mid x \geq t) = \phi(t,t) = r$. Following the approach of Section 3, we derive subsequently that

$$E_{[a,b]} = \{(\phi(x,t), 1_{\{x \geq t\}}) \quad | \quad a \leq x \leq b\}, \tag{56}$$

and that, even though the indicator function is not continuous,

$$e_{[a,b]}(c_1) = \frac{c_1}{r}. \tag{57}$$

The function $e_{[a,b]}(\cdot)$ corresponds to the dotted line in Figure 3. Since $e_{[a,b]}$ is proper, we have by Lemma 3.1 that

$$\tilde{v}_{[a,b]}(c_1) = e_{[a,b]}(c_1), \tag{58}$$

and by Lemma 2.6 that

$$\tilde{v}_{[a,b]}(c_1) = v(c_1, 1). \tag{59}$$

We have thus found the value of problem $v(c_1, 1)$.

6.4 Example 4: Maximal Tail-Value-at-Risk

For a given random variable X with distribution function F, we define its Tail-Value-at-Risk (TVaR) at probability level p by

$$\mathrm{TVaR}_p[X] := \inf_{t\in[a,b]} \left(t + \frac{1}{1-p} \int_{a-}^{b} (x-t)_+ dF(x) \right), \qquad p \in [0,1). \tag{60}$$

In many practical situations, no a priori probability measure will be given. When only partial (i.e., incomplete) information on the random variable is available, for example its mean, one may restrict oneself to the set of "admissible" distribution functions, satisfying the constraints implied by the information available, and then maximize the risk measure over this set. Doing so, one obtains what is often called a "worst case risk measurement", which can be regarded as a prudent assessment of the risk.

In the setting of Example 1, for a given number c_1, with $a < c_1 < b$, we consider the following primal problem:

$$v(c_1, 1) = \sup_{F\in\mathcal{G}} \left(\inf_{t\in[a,b]} \left(t + \frac{1}{1-p} \int_{a-}^{b} (x-t)_+ dF(x) \right) \right. \\ \left. \mid \int_{a-}^{b} x dF(x) = c_1, \int_{a-}^{b} dF(x) = 1 \right). \tag{61}$$

From the analysis in Example 1, it has become apparent that the 2-atomic extremal distribution solving the sup-problem for a given $t \in [a,b]$ does not depend on t, implying that one can interchange sup and inf in the primal problem above (maximin is minimax). Upon interchanging sup and inf, the value of the sup-problem at its solution can be derived analytically (see Example 1) and is given by

$$v(c_1, 1) = t + \frac{1}{1-p} (c_1 - a) \frac{b-t}{b-a}. \tag{62}$$

To derive the extremal TVaR, $v(c_1, 1)$ must be minimized with respect to t. Because the right-hand side of (62) is linear in t, we obtain a boundary solution, and the extremal TVaR is given by

$$\mathrm{TVaR}_p^e = \begin{cases} a + \frac{1}{1-p}(c_1 - a), & \frac{1}{1-p}\frac{c_1-a}{b-a} < 1; \\ b, & \frac{1}{1-p}\frac{c_1-a}{b-a} \geq 1. \end{cases}$$

6.5 Example 5: Maximal exponential premiums

In the setting of Example 1, for given numbers c_1, γ, with $a < c_1 < b$ and $\gamma > 0$, we consider the following primal problem (Gerber, 1974, Goovaerts *et al.*, 2004, Goovaerts, Kaas & Laeven, 2010a):

$$v(c_1, 1) = \sup_{F \in \mathcal{G}} \left(\int_{a-}^{b} \exp(x/\gamma) dF(x) \quad | \quad \int_{a-}^{b} x dF(x) = c_1, \int_{a-}^{b} dF(x) = 1 \right). \tag{63}$$

Equivalently, one may consider $\gamma \log(v(c_1, 1))$, which may be viewed as a special case of the family of risk measures introduced by Laeven & Stadje (2010).

It is not difficult to verify that the dual problem $\tilde{v}(c_1, 1)$ is given by

$$\tilde{v}(c_1, 1) = \inf_{\mathbf{d} \in \mathbb{R}^2} (d_1 c_1 + d_2 \quad | \quad d_1 x + d_2 \geq \exp(x/\gamma), \quad a \leq x \leq b). \tag{64}$$

Notice that the problem in (64) has the particular form of the special dual problem defined in (11), with $n = 2$, $\Theta = \Omega = [a, b]$, $f_\Theta(x) = f(x) = \exp(x/\gamma)$ and $g_{\Theta,1}(x) = g_1(x) = x$.

To determine the value $v(c_1, 1)$, we follow the approach of Section 3. The set $E_{[a,b]}$ is given by

$$E_{[a,b]} = \{(x, \exp(x/\gamma)) \quad | \quad a \leq x \leq b\}, \tag{65}$$

and corresponds to the bold line in Figure 4. The convex hull Co $\left[E_{[a,b]}\right]$ corresponds to the "smile shape" in Figure 4. Furthermore, $e_{[a,b]}(\cdot)$ corresponds to the dotted line in Figure 4.

Since $e_{[a,b]}$ is proper, we have by Lemma 3.1 that

$$\tilde{v}_{[a,b]}(c_1) = e_{[a,b]}(c_1) = (c_1 - a)\frac{\exp(b/\gamma) - \exp(a/\gamma)}{b - a} + \exp(a/\gamma), \tag{66}$$

and by Lemma 2.6 that

$$\tilde{v}_{[a,b]}(c_1) = v(c_1, 1). \tag{67}$$

We have thus found the value of problem $v(c_1, 1)$.

To find the function F that solves the primal problem (63), we proceed as follows: Lemma 3.2 implies that if $l_\mathbf{d}$ is an affine majorant of $e_{[a,b]}$ with $l_\mathbf{d}(c_1) = e_{[a,b]}(c_1)$, then $\mathbf{d}$ is a solution to the problem $\tilde{v}_{[a,b]}(c_1)$. Hence, we solve

$$d_1 c_1 + d_2 = (c_1 - a)\frac{\exp(b/\gamma) - \exp(a/\gamma)}{b - a} + \exp(a/\gamma), \tag{68}$$

for d_1, d_2. This gives the unique solution

$$d_1 = \frac{\exp(b/\gamma) - \exp(a/\gamma)}{b - a}, \qquad d_2 = -a\frac{\exp(b/\gamma) - \exp(a/\gamma)}{b - a} + \exp(a/\gamma). \tag{69}$$

Notice that for these particular values of d_1, d_2, the function $l_{\mathbf{d}}$ is an affine majorant of $e_{[a,b]}$.

The existence of a 2-atomic function F that solves problem (63) is guaranteed by Lemma 2.12. The atoms of the function F (i.e., the set $\{x \in [a,b] \mid dF(x) > 0\}$) can be derived by solving the atoms equation

$$\frac{\exp(b/\gamma) - \exp(a/\gamma)}{b-a}x - a\frac{\exp(b/\gamma) - \exp(a/\gamma)}{b-a} + \exp(a/\gamma) = \exp(x/\gamma) \tag{70}$$

for x; see also (9). The roots are a and b. Finally, to find the masses corresponding to these atoms, we solve the integral constraints

$$p_a + p_b = 1, \qquad p_a a + p_b b = c_1, \tag{71}$$

where $p_a = F(a)$ and $p_b = F(b) - F(a)$. This gives

$$p_a = \frac{b - c_1}{b-a}, \qquad p_b = \frac{c_1 - a}{b-a}. \tag{72}$$

Notice that for the derived values of d_1, d_2 and p_a, p_b, the equality (ii) of Lemma 2.7 holds. This confirms that we have indeed found a solution.

Now, for given numbers c_1, γ, with $a < c_1 < b$ and $\gamma > 0$, let us consider the following primal problem:

$$v_{u(m)}(c_1, 1) = \sup_{F \in \mathcal{G}_{u(m)}} \left(\int_{a-}^{b} \exp(x/\gamma) dF(x) \;\;\Big|\;\; \int_{a-}^{b} x dF(x) = c_1, \int_{a-}^{b} dF(x) = 1 \right), \tag{73}$$

with $a < m < b$. It is straightforward to verify that

$$\tilde{f}(x) = \gamma |\exp(x/\gamma) - \exp(m/\gamma)|, \qquad \tilde{g}_1(x) = \frac{1}{2}|x-m|(x+m), \qquad \tilde{g}_2(x) = |x-m|. \tag{74}$$

Since all f, g_1, g_2 are continuous at m, the dual problem is given by

$$\begin{aligned} \tilde{v}_{u(m)}(c_1, 1) = \inf_{\mathbf{d} \in \mathbb{R}^2} \Big(d_1 c_1 + d_2 \;\;\Big|\;\; & d_1 \frac{1}{2}|x-m|(x+m) + d_2 |x-m| \\ & \geq \gamma |\exp(x/\gamma) - \exp(m/\gamma)|, \quad a \leq x \leq b \Big). \end{aligned} \tag{75}$$

Notice that (75) can be cast into the form of the special dual problem defined in (11), with $n = 2$, $\Theta = [a,b]$, $f_\Theta(x) = \frac{\gamma|\exp(x/\gamma) - \exp(m/\gamma)|}{|x-m|}$ and $g_{\Theta,1}(x) = \frac{1}{2}(x+m)$, where $\frac{0}{0} = 0$ by convention. To determine the value of $\tilde{v}_{u(m)}(c_1, 1)$, we follow the approach of Section 3. The set $E_{[a,b]}$ is given by

$$E_{[a,b]} = \left\{ \left(\frac{1}{2}(x+m), \frac{\gamma|\exp(x/\gamma) - \exp(m/\gamma)|}{|x-m|} \right) \;\;\Big|\;\; a \leq x \leq b \right\}. \tag{76}$$

The convex hull $\mathrm{Co}\left[E_{[a,b]}\right]$ can be determined easily. Since $e_{[a,b]}$ is proper, we have by Lemma 3.1 that

$$\begin{aligned}\tilde{v}_{[a,b]}(c_1) = e_{[a,b]}(c_1) = &\frac{\left(c_1 - \frac{1}{2}(a+m)\right)}{\frac{1}{2}(b-a)} \\ &\times \left(\frac{\gamma(\exp(b/\gamma) - \exp(m/\gamma))}{b-m} - \frac{\gamma(\exp(m/\gamma) - \exp(a/\gamma))}{m-a}\right) \\ &+ \frac{\gamma(\exp(m/\gamma) - \exp(a/\gamma))}{m-a},\end{aligned} \tag{77}$$

and by Lemma 6.3 that

$$\tilde{v}_{[a,b]}(c_1) = v_{u(m)}(c_1, 1). \tag{78}$$

We have thus found the value of problem $v_{u(m)}(c_1, 1)$.

Extensions of the results derived in Examples 4 and 5 to cover Haezendonck risk measures with non-decreasing convex generator, and mean value premiums with non-decreasing convex disutility function are straightforward.

6.6 Example 6: Maximal stop-loss premium for the sum of two risks

Let $\Omega = [a,b]^2$, $a < b$, and let $\mathcal{A}$ be the Borel σ-algebra on Ω. We consider the convex cone $\mathcal{M}$ of countably additive, non-negative and finite measures on $\mathcal{A}$, including the atomic ones. With a given measure $\mu \in \mathcal{M}$, we identify the function $F : [a,b]^2 \to \mathbb{R}_+$ as follows:

$$F(x,y) = \mu([a,x] \times [a,y]), \qquad (x,y) \in [a,b]^2. \tag{79}$$

Furthermore, we define $F_1, F_2 : [a,b] \to \mathbb{R}_+$ by

$$F_1(x) = F(x,b), \qquad F_2(x) = F(b,x), \qquad x \in [a,b]. \tag{80}$$

Let $\mathcal{G}$ be the set of functions $F : [a,b]^2 \to \mathbb{R}_+$, which correspond one-to-one to an element in the set $\mathcal{M}$. Then, for given functions $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}_+$ and a given number t, with $2a < t < 2b$, we consider the following primal problem:

$$\begin{aligned}v(\varphi_1, \varphi_2, 1) = \sup_{F \in \mathcal{G}} \Bigg(\int_{a-}^{b} \int_{a-}^{b} (x+y-t)_+ dF(x,y) \quad | \quad &\int_{a-}^{b} (x-s_1)_+ dF_1(x) = \varphi_1(s_1), s_1 \in \mathbb{R}, \\ \int_{a-}^{b} (y-s_2)_+ dF_2(y) = \varphi_2(s_2), s_2 \in \mathbb{R}, &\int_{a-}^{b}\int_{a-}^{b} dF(x,y) = 1 \Bigg).\end{aligned}$$

That is, we aim to find the bivariate distribution function that maximizes the stop-loss premium of the sum of two risks, when the marginal stop-loss premiums are fixed and given. We note that if all stop-loss premiums of a distribution function are known, the distribution function itself is known as well. We note furthermore that there is an infinite number of integral constraints in this primal problem. The dual problem $\tilde{v}(\varphi_1, \varphi_2, 1)$ is then given by

$$\tilde{v}(\varphi_1, \varphi_2, 1) = \inf_{w_1, w_2, s_3} \left(\int_{\mathbb{R}} \varphi_1(s_1) dw_1(s_1) + \int_{\mathbb{R}} \varphi_2(s_2) dw_2(s_2) + s_3 \quad | \quad \int_{\mathbb{R}} (x - s_1)_+ dw_1(s_1) \right.$$
$$\left. + \int_{\mathbb{R}} (y - s_2)_+ dw_2(s_2) + s_3 \geq (x + y - t)_+, \quad a \leq x, y \leq b \right),$$

in which w_1 and w_2 are real-valued functions, and s_3 is a real number. This dual problem has the form of the special dual problem, though with infinite sums in both its objective function and its constraint specification. Hence, following the approach described in Section 3, we would have to resort to a discretization of the special dual problem. However, the dual formulation allows a simplification of the problem: notice that w_1, w_2 and s_3 must be non-negative and that w_1 and w_2 must satisfy $\int dw_i \geq 1$, $i = 1, 2$. Moreover, one can verify that because $\varphi_1(s_1)$ and $\varphi_2(s_2)$ are convex in s_1 and s_2, respectively, the solution to the dual problem must involve two 1-atomic distributions w_1 and w_2. Hence, the dual problem can be simplified to

$$\tilde{v}(\varphi_1, \varphi_2, 1) = \inf_{s_1, s_2, s_3} \left(\varphi_1(s_1) + \varphi_2(s_2) + s_3 \quad | \quad (x - s_1)_+ + (y - s_2)_+ + s_3 \geq (x + y - t)_+, \right.$$
$$\left. a \leq x, y \leq b \right).$$

Notice that in the above problem s_1, s_2 must satisfy $s_1 + s_2 \leq t$. Hence, we find that at the solution $s_1 + s_2 = t$ and $s_3 = 0$.

To solve

$$\tilde{v}(\varphi_1, \varphi_2, 1) = \inf_{s_1} \left(\varphi_1(s_1) + \varphi_2(t - s_1) \right),$$

we proceed as follows:

$$\frac{\partial}{\partial s_1} \left(\varphi_1(s_1) + \varphi_1(t - s_1) \right) = F_1(s_1) - F_2(t - s_1).$$

Next, equating to 0 and setting $p = F_1(s_1)$, we obtain

$$p - F_2(t - F_1^{-1}(p)) = 0,$$

and hence

$$t = F_1^{-1}(p) + F_2^{-1}(p),$$

where, as usual, we denote by F_i^{-1} the generalized inverse of F_i. We thus find that

$$\tilde{v}(\varphi_1, \varphi_2, 1) = \varphi_1(F_1^{-1}(p)) + \varphi_2(F_2^{-1}(p)).$$

Kaas *et al.* (2002) proved that when the two random variables X and Y are *comonotonic*, then for all points (x, y) in the support of (X, Y) and (s_1, s_2) being the point of intersection of the connected support of (X, Y) with the hyperplane $\{(s_1, s_2) | s_1 + s_2 = t\}$,

$$(x - s_1)_+ + (y - s_2)_+ = (x + y - t)_+.$$

We thus find that the maximal stop-loss premium for the sum of two risks is attained when the risks are comonotonic.

We refer the reader to Dhaene & Goovaerts (1996) for an alternative proof that comonotonic random variables have the largest stop-loss premiums. Generalizations to non-decreasing supermodular functions $\psi : \mathbb{R}^2 \to \mathbb{R}$ rather than $\psi = +$ (see also Theorem 3.1 of Laeven, 2009), and to a setting where only some moments of the marginal distributions are available are feasible. The latter problems are connected to mass transportation problems with partial knowledge of the marginals (MTPP).

6.7 Example 7: Maximal tail probability for the sum of two risks

Let again $\Omega = [a, b]^2$, and let $\mathcal{A}$ be the Borel σ-algebra on Ω. We consider the convex cone $\mathcal{M}$ of countably additive, non-negative and finite measures on $\mathcal{A}$, including the atomic ones. For a given measure $\mu \in \mathcal{M}$, let $F : [a, b]^2 \to \mathbb{R}_+$ be as defined in (79), and let $F_1, F_2 : [a, b] \to \mathbb{R}_+$ be as defined in (80). Furthermore, let $\mathcal{G}$ be as defined in the previous example. Then, for given functions $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}_+$ and a given number t, with $2a < t < 2b$, we consider the following primal problem:

$$v(\varphi_1, \varphi_2, 1) = \sup_{F \in \mathcal{G}} \left(\int_{a-}^{b} \int_{a-}^{b} 1_{\{x+y \geq t\}} dF(x, y-) \quad | \quad \int_{a-}^{b} 1_{\{x > s_1\}} dF_1(x) = \varphi_1(s_1), s_1 \in \mathbb{R}, \right.$$
$$\left. \int_{a-}^{b} 1_{\{y > s_2\}} dF_2(y) = \varphi_2(s_2), s_2 \in \mathbb{R}, \int_{a-}^{b} \int_{a-}^{b} dF(x, y) = 1 \right),$$

with $F(\cdot, \cdot -)$ the version of F that is left-continuous in its second coordinate. We thus aim to find the bivariate distribution function that has a maximal (bivariate) tail probability, when the marginal tail probabilities (and hence the marginal distributions) are fixed and given.

Notice that, in contrast to what is common, we use a left-continuous tail probability in the objective function. It will guarantee that the upper bound is best-possible, i.e., attained (see also Kaas, Laeven & Nelsen, 2009, and the references therein). Notice furthermore that the problem of finding the maximal (i.e., worst) tail probability is equivalent to the problem of finding the maximal (i.e., worst) Value-at-Risk. That is, if the maximal tail probability is known for all values t, then also the maximal Value-at-Risk is known for all percentile levels, and vice versa.

The dual problem $\tilde{v}(\varphi_1, \varphi_2, 1)$ is then given by

$$\tilde{v}(\varphi_1, \varphi_2, 1) = \inf_{w_1, w_2, s_3} \left(\int_{\mathbb{R}} \varphi_1(s_1) dw_1(s_1) + \int_{\mathbb{R}} \varphi_2(s_2) dw_2(s_2) + s_3 \quad | \quad \int_{\mathbb{R}} 1_{\{x>s_1\}} dw_1(s_1) \right.$$
$$\left. + \int_{\mathbb{R}} 1_{\{y>s_2\}} dw_2(s_2) + s_3 \geq 1_{\{x+y\geq t\}}, \quad a \leq x, y \leq b \right),$$

in which w_1 and w_2 are real-valued functions, and s_3 is a real number.

As in the previous example, the dual formulation allows a simplification of the problem (cf. Rachev & Rüschendorf, 1998, Section 3.5):

$$\tilde{v}(\varphi_1, \varphi_2, 1) = \inf_{s_1, s_2, s_3} \big(\varphi_1(s_1) + \varphi_2(s_2) + s_3 \quad | \quad 1_{\{x>s_1\}} + 1_{\{y>s_2\}} + s_3 \geq 1_{\{x+y\geq t\}},$$
$$a \leq x, y \leq b \big).$$

Hence, we find that:

$$\tilde{v}(\varphi_1, \varphi_2, 1) = \min \left\{ 1; \inf_{s_1} \big(\varphi_1(s_1) + \varphi_2^-(t - s_1) \big) \right\},$$

with φ_2^- the left-continuous version of φ_2.

The derived bound remains valid when $n > 2$ but in that case it is no longer best-possible.

6.8 Example 8: Maximal tail probability for the sum of two risks under partial information on the dependence structure

In this example, we consider the problem of finding the maximal tail probability for the sum of two risks, when the marginals are known but the dependence structure is only partially known. In particular, we assume that the dependence structure can be represented by a mixture of copulas that belong to one parametric copula family, e.g., Clayton, Frank or Gumbel. In addition, we assume that the value of Spearman's measure of concordance is known. However, the particular mixture distribution that is used to generate the mixture of copulas is supposed to be unknown. The mixture distribution determines the latent randomness in the model and is sometimes referred to as a *structure distribution*.

We recall that a bivariate copula $C : [0,1]^2 \to [0,1]$ is a bivariate distribution with uniform-$(0,1)$ marginals. Furthermore, we recall that for a given copula C, Spearman's rho, denoted by ρ_C, is defined by

$$\rho_C := 12 \int_0^1 \int_0^1 xy dC(x,y) - 3. \tag{81}$$

It is well-known (see e.g., Nelsen (1999), Section 3.2.4) that if for some set Ω, $\{C(x,y;\theta), \theta \in \Omega\}$ denotes a parametric family of bivariate copulas and $F : \Omega \to [0,1]$ denotes a mixture distribution, then the convex sum $\int_\Omega C(x,y;\theta) dF(\theta)$ is again a copula.

In the following, we let $\Omega = [a,b]$, $a < b$, and let $\mathcal{A}$ be the Borel σ-algebra on Ω. Furthermore, we let $\mathcal{M}$, F and $\mathcal{G}$ be as defined in Example 1. Then, for given marginals F_1, F_2 (which are here unrelated to F; this in contrast to the previous two examples in which F denoted a bivariate distribution function with marginals F_1 and F_2) and given numbers $t \in \mathbb{R}$ and c_1, with $\frac{1}{6} < c_1 < \frac{1}{3}$, we state the following primal problem:

$$\begin{aligned} v(c_1, 1) = \sup_{F \in \mathcal{G}} \Bigg(& \int_{a-}^{b} \left(\int_0^1 \int_0^1 1_{\{F_1^{-1}(x) + F_2^{-1}(y) \geq t\}} dC(x,y;\theta) \right) dF(\theta) \quad | \\ & \int_{a-}^{b} \left(\int_0^1 \int_0^1 xy dC(x,y;\theta) \right) dF(\theta) = c_1, \int_a^b dF(\theta) = 1 \Bigg), \end{aligned} \tag{82}$$

Then, the dual problem is given by

$$\begin{aligned} \tilde{v}(c_1, 1) = \int_{d \in \mathbb{R}^2} \Bigg(& d_1 c_1 + d_2 \quad | \quad d_1 \left(\int_0^1 \int_0^1 xy dC(x,y;\theta) \right) + d_2 \\ & \geq \left(\int_0^1 \int_0^1 1_{\{F_1^{-1}(x) + F_2^{-1}(y) \geq t\}} dC(x,y;\theta) \right), \quad a \leq \theta \leq b \Bigg), \end{aligned}$$

which has the form of the special dual problem. Hence, one can easily verify that the set $E_{[a,b]}$ is given by

$$E_{[a,b]} = \left\{ \left(\left(\int_0^1 \int_0^1 xy dC(x,y;\theta) \right), \left(\int_0^1 \int_0^1 1_{\{F_1^{-1}(x) + F_2^{-1}(y) \geq t\}} dC(x,y;\theta) \right) \right) \quad | \quad a \leq \theta \leq b \right\}.$$

As a simple example, we consider the Farlie-Gumbel-Morgenstern family of copulas (see e.g., Nelsen (1999), p. 68), defined by

$$C^{\text{FGM}}(x,y;\theta) = xy + \theta xy(1-x)(1-y), \qquad -1 \leq \theta \leq 1.$$

It is straightforward to verify that for this family of copulas, Spearman's rho is given by $\rho_C = \frac{\theta}{3}$, thus is restricted to the interval $[-\frac{1}{3}, \frac{1}{3}]$. Furthermore, we derive that for uniform-$(0,1)$ marginals F_1 and F_2, and given numbers t and θ

$$\int_0^1 \int_0^1 1_{\{F_1^{-1}(x)+F_2^{-1}(y)\geq t\}} dC^{\mathrm{FGM}}(x,y;\theta)$$
$$= \begin{cases} \int_0^1 \int_{\max(t-x,0)}^1 (1+\theta(1-2x)(1-2y))\,dxdy, & 0\leq t\leq 1, \\ \int_{t-1}^1 \int_{\min(t-x,1)}^1 (1+\theta(1-2x)(1-2y))\,dxdy, & 1\leq t\leq 2, \end{cases}$$
$$= \begin{cases} 1-\frac{1}{2}t^2-\frac{1}{2}\theta t^2+\frac{2}{3}\theta t^3-\frac{1}{6}\theta t^4, & 0\leq t\leq 1, \\ 2-2t+\frac{1}{2}t^2-\frac{2}{3}\theta+\frac{2}{3}\theta t+\frac{1}{2}\theta t^2-\frac{2}{3}\theta t^3+\frac{1}{6}\theta t^4, & 1\leq t\leq 2. \end{cases}$$

Hence, we can determine the set $E_{[-1,1]}$ and its convex hull. For the simple example under consideration, we find that $E_{[-1,1]}$ is a straight line, hence coincides with $e_{[-1,1]}(\cdot)$. Since $e_{[-1,1]}(\cdot)$ is proper, we have by Lemma 3.1 that for $-\frac{1}{3}\leq c_1\leq\frac{1}{3}$

$$\tilde{v}_{[-1,1]}(c_1)=e_{[-1,1]}(c_1)=\begin{cases} 1-\frac{1}{2}t^2-\frac{3}{2}c_1t^2+2c_1t^3-\frac{1}{2}c_1t^4, & 0\leq t\leq 1, \\ 2-2t+\frac{1}{2}t^2-2c_1+2c_1t+\frac{3}{2}c_1t^2-2c_1t^3+\frac{1}{2}c_1t^4, & 1\leq t\leq 2. \end{cases}$$

7 Conclusion

We have illustrated that the dual approach is a powerful tool for solving problems of worst case risk measurement. Many such problems can be cast into the form of the special dual problem. Solving the special dual problem amounts to determining the convex hull of the set E_Θ. When the number of integral constraints (including the constraint that $\int d\mu = 1$) gets larger than three, the visual geometry of the dual approach disappears and the determination of the convex hull of E_Θ becomes somewhat more complex. However, in that case we can resort to well-defined numerical methods to solve the special dual problem. Even extensions to non-linear forms instead of $\int f d\mu$ have been considered by Etienne De Vijlder. Examples 6.6, 6.7 and 6.8 have demonstrated that the dual approach can also be applied in a multivariate setting.

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Figure 1: Geometric illustration of Example 1: Maximal stop-loss premiums

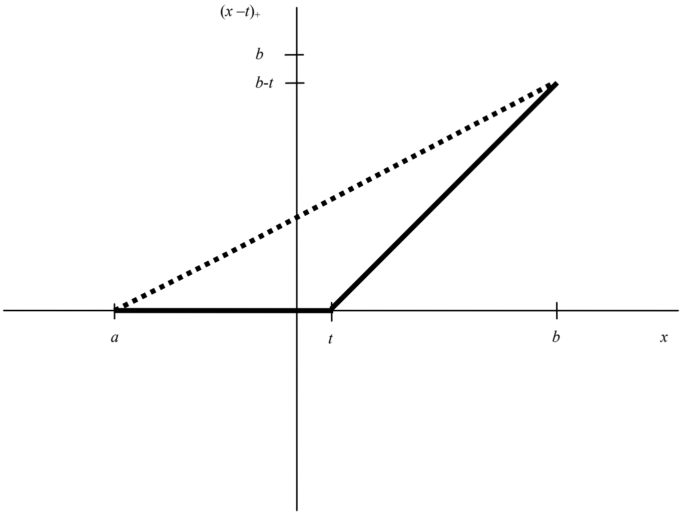

Figure 2: Geometric illustration of Example 2: Maximization over a cone of m-unimodal measures

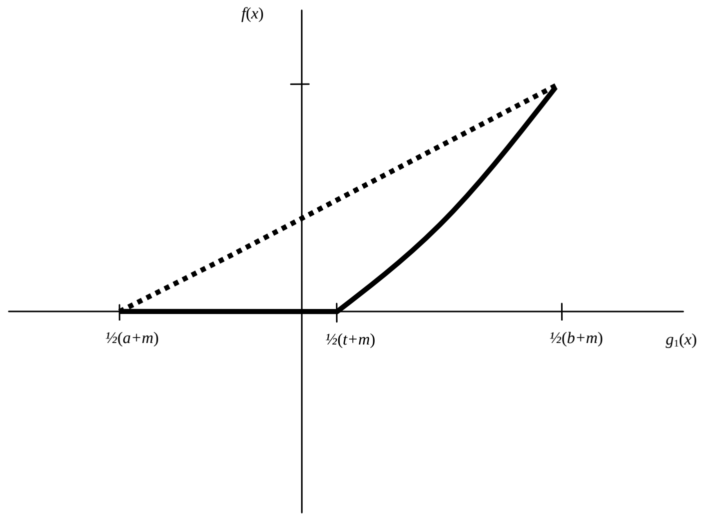

Figure 3: Geometric illustration of Example 3: Maximal tail probability

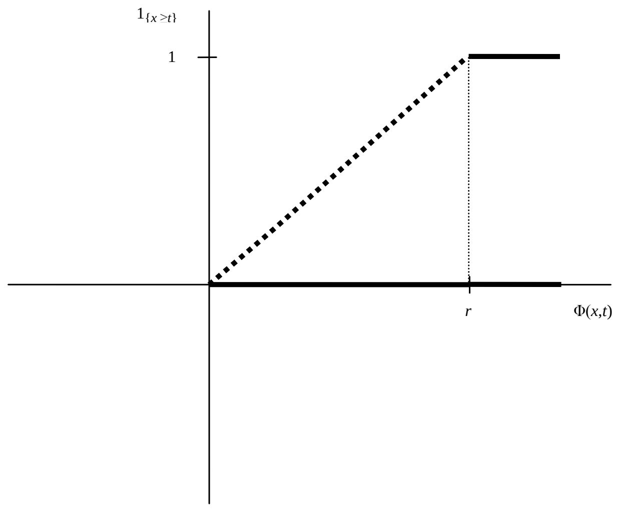

Figure 4: Geometric illustration of Example 5: Maximal exponential premiums