

Actuarial Risk Measures for Financial Derivative Pricing

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Abstract

We present an axiomatic characterization of price measures that are superadditive and comonotonic additive for normally distributed random variables. The price representation derived, involves a probability measure transform that is closely related to the Esscher transform, and we call it the *Esscher-Girsanov transform*. In a financial market in which the primary asset price is represented by a stochastic differential equation with respect to Brownian motion, the price mechanism based on the Esscher-Girsanov transform can generate approximate-arbitrage-free financial derivative prices.

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1 Introduction

Risk measures for actuarial pricing are usually justified by means of an axiomatic characterization; see e.g., Goovaerts, De Vylder & Haezendonck (1984). Financial derivative pricing usually relies on principles of no arbitrage. Various attempts to connect the two approaches are available in the literature; the interested reader is referred to Embrechts (2000) for a review. This paper establishes a new connection.

The connection is based on the time-honored *Esscher transform*. The Esscher transform has proven to be a valuable tool for the pricing of insurance and financial products. In Bühlmann (1980), a premium principle based on the Esscher transform is derived within a general equilibrium model in which decision makers have negative exponential utility functions; see Iwaki, Kijima & Morimoto (2001) for an extension of that model to a multi-period setting. Gerber & Goovaerts (1981) established an axiomatic characterization of an additive premium principle that involves a mixture of Esscher transforms.

In a financial environment, Gerber & Shiu (1994, 1996) use the Esscher transform to construct equivalent martingale measures for Lévy processes (with independent and stationary increments). Inspired by this, Bühlmann *et al.* (1996) more generally use *conditional Esscher transforms* to construct equivalent martingale measures for classes of semi-martingales.

In this paper, the actuarial approach of establishing risk evaluation mechanisms by means of an axiomatic characterization is used to characterize a pricing mechanism that can generate approximate-arbitrage-free financial derivative prices. In particular, this paper presents a representation theorem for price measures that are superadditive and comonotonic additive for normally distributed random variables. The price representation derived involves a probability measure transform that is closely related to the Esscher transform. We demonstrate that in a financial market in which the primary asset price is represented by a stochastic differential equation with respect to Brownian motion, approximate-arbitrage-free financial derivative prices coincide with the price representation derived.

The axioms imposed to establish the representation theorem can be formulated as follows:

1. Ordered *Esscher-Girsanov transforms* implies ordered prices. If the price measure is applied to normally distributed random variables, this axiom is equivalent to “respect for second-order stochastic dominance”.
2. The price measure is appropriately normalized such that the price of c non-random

units is equal to c non-random units.

3. Additivity for sums of *Esscher-Girsanov transforms*. If the price measure is applied to normally distributed random variables, this axiom is equivalent to “superadditivity and comonotonic additivity of the price measure”.
4. Continuity conditions, which are necessary to establish the mathematical proofs.

The outline of this paper is as follows: in Section 2, we consider the Esscher transform, we establish some stochastic order relations derived from it and we discuss the axiomatization of the mixed Esscher principle. In Section 3, we introduce the *Esscher-Girsanov transform* and axiomatize a price measure induced by it. Section 4 addresses the pricing of financial derivatives by means of Esscher-Girsanov transforms.

2 Stochastic Ordering and the Esscher Transform

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this paper, unless otherwise stated, a random variable (r.v.) represents net income or profit at a future point in time. Throughout, we assume that for any r.v. defined on the probability space, its *moment generating function* exists, i.e., for any r.v. $X : \Omega \rightarrow \mathbb{R}$

$$\mathbb{E}[e^{hX}] < +\infty, \quad h \in \mathbb{R}. \quad (1)$$

For the cumulative distribution function (cdf) $F_X(\cdot)$ with differential $dF_X(\cdot)$, corresponding to a given r.v. X , we define by

$$dF_X^{(h)}(x) = \frac{e^{hx} dF_X(x)}{\mathbb{E}[e^{hX}]}, \quad h \in \mathbb{R}, \quad (2)$$

its *Esscher transform* with parameter h . Esscher (1932) suggested to use the transform in (2) instead of the original cdf, to apply the well-known Edgeworth approximation to; see also Gerber (1979). The reason was that the Edgeworth approximation performs well in the vicinity of the expectation, but performs worse in the tails. Notice that for $h = 0$, the original differential appears, and that $F_X(\cdot)$ and its Esscher transform $F_X^{(h)}(\cdot)$ are *equivalent distributions* in the sense that they have the same null sets. It is not difficult to verify that for a normal cdf with expectation μ and variance σ^2 , its Esscher transform is a normal cdf with expectation $\mu + h\sigma^2$ and variance σ^2 .

Next, for a given r.v. X , we define the real-valued function $\psi_X(\cdot)$ as follows:

$$\psi_X(h) = \int_{-\infty}^{+\infty} x dF_X^{(h)}(x) = \frac{\mathbb{E}[Xe^{hX}]}{\mathbb{E}[e^{hX}]}. \quad (3)$$

The number $\psi_X(h)$ is known as the *Esscher premium* with parameter h ; see Bühlmann (1980) and Goovaerts, De Vylder & Haezendonck (1984). Notice that $\psi_X(h)$ is non-decreasing in h . This can be proved easily by the Hölder-inequality and will be used later.

In the following, we denote by the functional $\pi[\cdot]$ a *risk measure* or –since X is interpreted as net income or profit– rather *price measure* that assigns a real number to any r.v. or its cdf. We introduce a set of axioms that $\pi[\cdot]$ must satisfy:

- A1. If $\psi_X(h) \leq \psi_Y(h)$ for all $h \leq 0$, then $\pi[X] \leq \pi[Y]$;
- A2. $\pi[c] = c$, for all c ;
- A3. $\pi[X + Y] = \pi[X] + \pi[Y]$ when X and Y are independent;
- A4. If X_n converges weakly to X , with $\min[X_n] \rightarrow \min[X]$, then $\lim_{n \rightarrow +\infty} \pi[X_n] = \pi[X]$.

In a general setting, axiom A1 can be criticized. Gerber (1981) already pointed out that the Esscher premium is not monotonic, i.e., it does not hold that if X is first-order stochastically dominated by Y , denoted by $X \leq_{st} Y$, then $\psi_X(h) \leq \psi_Y(h)$ for all $h \in \mathbb{R}$ (or even all $h \leq 0$). Hence, axiom A1 does not guarantee monotonicity of the functional $\pi[\cdot]$.

Goovaerts *et al.* (2004) replaced axiom A1 by the more restrictive axiom of respect for *Laplace transform* order, which does guarantee monotonicity of the functional $\pi[\cdot]$. We say that X is smaller than Y in Laplace transform order if $\mathbb{E}[e^{hX}] \geq \mathbb{E}[e^{hY}]$ for all $h \leq 0$. We write $X \leq_{Lt} Y$. Indeed, $X \leq_{st} Y$ implies $X \leq_{Lt} Y$. In the expected utility model, the Laplace transform order represents preferences of decision makers with a *negative exponential* utility function given by

$$U(x) = 1 - e^{hx}, \quad h < 0. \quad (4)$$

Here, $-h$ is the Arrow-Pratt measure of absolute risk aversion. The interested reader is referred to Denuit (2001) for a comprehensive treatment of the Laplace transform order.

In the next sections, normally distributed r.v.'s are of particular interest. Suppose that X and Y are normally distributed. Then the condition that

$$\psi_X(h) \leq \psi_Y(h), \quad h \leq 0, \quad (5)$$

is equivalent to the condition that both $\mu_X \leq \mu_Y$ and $\sigma_X \geq \sigma_Y$. To verify this statement, notice that for normally distributed r.v.'s

$$\psi_X(h) = \mu_X + h\sigma_X^2. \quad (6)$$

Furthermore, it is not difficult to verify that if X and Y are normally distributed, then $X \leq_{Lt} Y$ if and only if condition (5) is satisfied (or equivalently if and only if both $\mu_X \leq \mu_Y$ and $\sigma_X \geq \sigma_Y$).

More generally, it is well-known that if X and Y are normally distributed with $\mu_X \leq \mu_Y$ and $\sigma_X \geq \sigma_Y$, then X is second-order stochastically dominated by Y and hence Y is preferred to X by any risk averse expected utility decision maker (with concave utility function); see Hadar & Russell (1969) and Rothschild & Stiglitz (1970) for the original work on second-order stochastic dominance. In particular, notice that for normally distributed r.v.'s, $X \leq_{st} Y$ if and only if $\mu_X \leq \mu_Y$ and $\sigma_X = \sigma_Y$. Hence, axiom A1 is appealing for the case of normally distributed r.v.'s.

In the economic literature, axiom A2 is often referred to as the *certainty equivalent* condition. Notice that c plays two roles in axiom A2: a r.v. degenerated at c on the left-hand side and a real number on the right-hand side.

The desirability of price additivity for independent r.v.'s, as imposed by axiom A3, was already pointed out by Borch (1962), p. 429; see also Bühlmann (1985).

Axiom A4 is a continuity condition on the price measure $\pi[\cdot]$. We state the following lemma:

Lemma 2.1 *A price measure $\pi[\cdot]$ satisfies the set of axioms A1-A4 if and only if there exists some non-decreasing function $H : [-\infty, 0] \rightarrow [0, 1]$ such that*

$$\pi [X] = \int_{[-\infty, 0]} \psi_X(h) dH(h). \quad (7)$$

Proof: The proof of this lemma can be established along the same lines as the proof of Theorem 2 of Gerber & Goovaerts (1981); see Goovaerts *et al.* (2004) for comments on that proof. \square

Some remarks:

Remark 2.1 *Gerber & Goovaerts (1981) established an axiomatic characterization of the mixed Esscher principle. Goovaerts et al. (2004) axiomatized the mixed exponential principle. It is straightforward to verify that for normally distributed r.v.'s any mixed Esscher premium is a mixed exponential premium, and vice versa. In general, it only holds that any mixed exponential premium is a mixed Esscher premium; see Goovaerts et al. (2004).* \square

Remark 2.2 *The mixture function $H(\cdot)$ can be regarded as a cdf, supported on $(-\infty, 0]$ and possibly defective with a jump at $-\infty$. It can serve as a prior distribution for the*

Arrow-Pratt measure of absolute risk aversion; see in this respect Savage (1954). To see why the parameter $-h$ involved in the Esscher transform can be interpreted as the Arrow-Pratt measure of absolute risk aversion corresponding to a decision maker with a negative exponential utility function, we refer to Goovaerts, De Vylder & Haezendonck (1984), pp. 84-86. \square

Remark 2.3 The price measure $\pi[\cdot]$ characterized in Lemma 2.1 can be expressed as $\pi[X] = \mathbb{E}^*[X]$, where the expectation is calculated using the differential

$$dF_X^{(H(\cdot))}(x) = \left(\int_{h \in [-\infty, 0]} \frac{e^{hx} dH(h)}{\mathbb{E}[e^{hX}]} \right) dF_X(x).$$

\square

3 The Esscher-Girsanov Transform

In the previous section, we presented a representation theorem for price measures that are additive for independent r.v.'s. The price representation derived can be regarded as an expectation under a (mixed) Esscher transformed probability measure. In this section, we introduce a closely related probability measure transform and axiomatize a price measure induced by it.

For a given r.v. X , we define the extended real-valued function $\phi_X(\cdot)$ as follows:

$$\phi_X(x) = \Phi^{-1}(F_X(x)), \tag{8}$$

in which $\Phi^{-1}(\cdot)$ denotes the inverse distribution function of the standard normal distribution. It is well-known that if F_X is continuous, then the r.v. $\phi_X(X)$ is normally distributed with mean 0 and variance 1. In the remainder of this section, unless otherwise stated, we restrict ourselves to r.v.'s with a continuous cdf. We state the following definition:

Definition 3.1 For the cdf $F_X(\cdot)$ with differential $dF_X(\cdot)$ corresponding to a given r.v. X , and a given real number v , we define by

$$dF_X^{(h,v)}(x) = \frac{e^{hv\phi_X(x)}}{\mathbb{E}[e^{hv\phi_X(X)}]} dF_X(x) = e^{hv\phi_X(x) - \frac{1}{2}h^2v^2} dF_X(x), \quad h \in \mathbb{R}, \tag{9}$$

its Esscher-Girsanov transform with parameters h and v (absolute risk aversion and penalty parameter, respectively). \square

The name of Igor V. Girsanov is attached to the probability measure transform defined above to emphasize the close resemblance between the Radon-Nikodym derivative used in (9) and the Radon-Nikodym derivative used in Girsanov's Theorem; see e.g., Karatzas & Shreve (1988).

It is not difficult to verify that for a normal cdf with expectation μ and variance σ^2 , its Esscher-Girsanov transform is a normal cdf with expectation $\mu + hv\sigma$ and variance σ^2 . In particular, if $v = \sigma$, we trivially find that the Esscher-Girsanov transform is an ordinary Esscher transform. Hence, for a normal cdf, the Esscher-Girsanov transform, just like the ordinary Esscher transform, changes the mean while preserving the variance. Notice that for the change of the mean, the value of the mean is irrelevant.

In the following, we let v be strictly positive and temporarily fixed and restrict the domain of h to $h \leq 0$.

We introduce the real-valued function $\psi_X^v(\cdot)$ defined by

$$\psi_X^v(h) = \int_{-\infty}^{+\infty} x dF_X^{(h,v)}(x) = \mathbb{E} \left[X e^{hv\phi_X(X) - \frac{1}{2}h^2v^2} \right], \quad h \leq 0. \quad (10)$$

Henceforth, the number $\psi_X^v(h)$ is called the *Esscher-Girsanov price* of the r.v. X , with parameters $h \leq 0$ and $v > 0$. Notice that given v , there exists a unique correspondence between X and its Esscher-Girsanov price in the sense that $X = Y$ in distribution if and only if

$$\psi_X^v(h) = \psi_Y^v(h), \quad h \leq 0. \quad (11)$$

To verify this statement, notice that

$$\psi_X^v(h) = \int_{-\infty}^{+\infty} F_X^{-1}(\Phi(y)) e^{hvy - \frac{1}{2}h^2v^2} d\Phi(y), \quad (12)$$

which can be regarded as a Laplace transform, so that the one-to-one correspondence between $\psi_X^v(\cdot)$ and $F_X^{-1}(\cdot)$ follows. The derivative of $\psi_X^v(h)$ with respect to h is given by

$$v \left(\frac{\mathbb{E} [X\phi_X(X)e^{hv\phi_X(X)}]}{\mathbb{E} [e^{hv\phi_X(X)}]} - \frac{\mathbb{E} [Xe^{hv\phi_X(X)}]}{\mathbb{E} [e^{hv\phi_X(X)}]} \frac{\mathbb{E} [\phi_X(X)e^{hv\phi_X(X)}]}{\mathbb{E} [e^{hv\phi_X(X)}]} \right), \quad (13)$$

in which the expression between brackets can be regarded as the *Esscher-Girsanov covariance* of X and $\phi_X(X)$ and is non-negative.

As was pointed out in Goovaerts *et al.* (2004), the price measure characterized in Lemma 2.1 has a counterpart that assigns a real number to the function $\psi_X(\cdot)$. Similarly, we denote by $\rho^v[\cdot]$ a functional that assigns a real number to any function $\psi_X^v(\cdot)$, we let the price measure $\pi^v[\cdot]$ be defined by

$$\pi^v[X] = \rho^v[\psi_X^v], \quad (14)$$

and state the following set of axioms that $\rho^v[\cdot]$ should satisfy:

B1. If $\psi_X^v(h) \leq \psi_Y^v(h)$ for all $h \leq 0$, then $\rho^v[\psi_X^v] \leq \rho^v[\psi_Y^v]$;

B2. $\rho^v[c] = c$, for all c ;

B3. $\rho^v[\psi_X^v + \psi_Y^v] = \rho^v[\psi_X^v] + \rho^v[\psi_Y^v]$;

B4. If $\psi_{X_n}^v(h)$ converges to $\psi_X^v(h)$ for all $h \in [-\infty, 0]$, then $\lim_{n \rightarrow +\infty} \rho^v[\psi_{X_n}^v] = \rho^v[\psi_X^v]$.

Notice that axioms B2 and B4 are similar to axioms A2 and A4. Notice furthermore that $\psi_{cX}^v(h) = c\psi_X^v(h)$ for all $c \geq 0$. Hence, axioms B3 and B4 imply that the price of a portfolio cX equals c times the price of X . This is an intuitive condition whenever financial markets are sufficiently liquid.

We note that for normally distributed r.v.'s, axiom B1 is similar to axiom A1 and gives rise to the appealing second-order stochastic dominance preserving property for $\pi^v[\cdot]$. One easily verifies that if X and Y are two normally distributed r.v.'s with linear correlation coefficient ρ_{XY} , then

$$\psi_{X+Y}^v(h) = \mu_X + \mu_Y + hv\sqrt{\sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2}. \quad (15)$$

Hence, for normally distributed r.v.'s, axiom B3 is equivalent to the condition that the price of the portfolio $X + Y$ is equal to the price of a normally distributed r.v. \tilde{X} with mean $\alpha(\mu_X + \mu_Y)$ and standard deviation $\beta\sqrt{\sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2}$ plus the price of a normally distributed r.v. \tilde{Y} with mean $(1 - \alpha)(\mu_X + \mu_Y)$ and standard deviation $(1 - \beta)\sqrt{\sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2}$, for any pair (α, β) with $0 \leq \alpha, \beta \leq 1$.

For later reference, we state the following equivalent definitions for a pair of r.v.'s to be *comonotonic*; we follow Denneberg (1994), Proposition 4.5:

Definition 3.2 *We say that a pair of r.v.'s $X, Y : \Omega \rightarrow \mathbb{R}$ is comonotonic, denoted by X^c, Y^c , if*

1. *there exists no pair ω_1, ω_2 such that $X(\omega_1) < X(\omega_2)$ while $Y(\omega_1) > Y(\omega_2)$;*
2. *there exists a r.v. $Z : \Omega \rightarrow \mathbb{R}$ and non-decreasing function $a(\cdot)$ and $b(\cdot)$ on \mathbb{R} such that*

$$X(\omega) = a(Z(\omega)), \quad Y(\omega) = b(Z(\omega)), \quad \text{for all } \omega \in \Omega.$$

□

It is well-known that for a bivariate normal comonotonic couple (X^c, Y^c) it holds that $\rho_{\tilde{X}^c \tilde{Y}^c} = 1$. Hence, using (15) and axioms B1 and B3, respectively, one easily verifies that for a bivariate normal couple (X, Y)

$$\rho^v[\psi_{X+Y}^v] \geq \rho^v[\psi_{X^c+Y^c}^v] = \rho^v[\psi_X^v] + \rho^v[\psi_Y^v], \quad (16)$$

recalling that $h \leq 0$ and that $v > 0$. This means that for normally distributed r.v.'s, axiom B3 is equivalent to superadditivity and comonotonic additivity of the price measure $\pi^v[\cdot]$, which captures the diversification benefit of pooling.

Then we state the following theorem:

Theorem 3.1 *A functional $\rho^v[\cdot]$ satisfies the set of axioms B1-B4 if and only if there exists some non-decreasing function $H : [-\infty, 0] \rightarrow [0, 1]$ such that*

$$\rho^v[\psi_X^v] = \int_{[-\infty, 0]} \psi_X^v(h) dH(h). \quad (17)$$

Proof: Just as the proof of Lemma 2.1, the proof of this theorem can be established along the same lines as the proof of Theorem 2 of Gerber & Goovaerts (1981); see again Goovaerts *et al.* (2004) for comments on that proof. \square

Since the next section will consider stochastic processes instead of r.v.'s, the definition of the Esscher-Girsanov transform has to be generalized. In the remainder of this section, we consider a discrete-time stochastic process $X = (X_i : i = 1, 2, \dots)$, $X_0 = x_0$, with independent increments. Clearly, it holds that

$$dF_{X_n|X_0}(x_n|x_0) = \int_{x_{n-1}} \cdots \int_{x_1} dF_{X_n|X_{n-1}}(x_n|x_{n-1}) dF_{X_{n-1}|X_{n-2}}(x_{n-1}|x_{n-2}) \cdots dF_{X_1|X_0}(x_1|x_0).$$

We state the following definition:

Definition 3.3 *For the cdf $F_{X_n}(\cdot)$ with differential $dF_{X_n}(\cdot)$ corresponding to a given continuous r.v. X_n , and a given strictly positive function $v(\cdot)$, we define by*

$$\begin{aligned} dF_{X_n|X_0}^{(h, v(\cdot))}(x_n|x_0) &= \int_{x_{n-1}} \cdots \int_{x_1} e^{h \sum_{j=0}^{n-1} v(x_j) \phi_{X_{j+1}|X_j}(x_{j+1}|x_j) - \frac{1}{2} h^2 v(x_j)^2} \\ &\quad \times dF_{X_n|X_{n-1}}(x_n|x_{n-1}) dF_{X_{n-1}|X_{n-2}}(x_{n-1}|x_{n-2}) \cdots dF_{X_1|X_0}(x_1|x_0) \end{aligned} \quad (18)$$

its discrete-time Esscher-Girsanov transform with parameter $h \leq 0$ and penalty function $v(\cdot)$.

□

The discrete-time Esscher-Girsanov transform can be regarded as a particular example of a *conditional Esscher transform* (see Bühlmann *et al.* (1996)), though there is a subtle difference being that, in accordance with the economic interpretation and axiomatization, we use a constant Arrow-Pratt measure of absolute risk aversion.

4 Financial Derivative Pricing by Means of Esscher-Girsanov Transforms

In this section, we will show that in a financial market in which the primary asset price is represented by a stochastic differential equation (SDE) with respect to Brownian motion, the price mechanism based on the Esscher-Girsanov transform can generate approximate-arbitrage-free financial derivative prices.

We consider a finite time horizon $T < +\infty$. The flow of information is represented by the completed and right continuous filtration $\mathbb{F} = (\mathcal{F}_t : 0 \leq t \leq T)$, with for all $s \leq t \leq T$, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T = \mathcal{F}$. Henceforth, for a given r.v. X , we denote by $\mathbb{E}_t[X] = \mathbb{E}[X|\mathcal{F}_t]$ the conditional expectation of X given \mathcal{F}_t .

We consider a time-homogeneous primary asset price process $S = (S_t : 0 \leq t \leq T)$, defined by a stochastic differential equation of the form

$$S_0 = s_0, \quad dS_t = \mu(S_t)dt + \sigma(S_t)dB_t, \quad (19)$$

in which $\mu : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ and $B = (B_t : 0 \leq t \leq T)$ denotes a standard Brownian motion. Henceforth, we understand $\sigma(S_t)dB_t$ in the usual “Itô sense” (i.e., left point discretization). Under well-known regularity conditions on $\mu(\cdot)$ and $\sigma(\cdot)$ there exists a unique Itô process S that solves (19) for each starting point s_0 .

Next, we consider a bond price process $\beta = (\beta_t : 0 \leq t \leq T)$, defined by the SDE

$$\beta_0 > 0, \quad d\beta_t = \beta_t r(S_t)dt, \quad (20)$$

in which $r : \mathbb{R} \rightarrow \mathbb{R}$ is sufficiently smooth for the existence of the integral

$$\exp \left[\int_0^t r(S_\tau) d\tau \right], \quad t \in (0, T]. \quad (21)$$

Although we restrict ourselves to time-homogeneous primary asset price processes, a generalization to general diffusion processes is feasible. Notice, however, that most of

the well-known diffusion processes (e.g., the (Geometric) Wiener process, the Ornstein-Uhlenbeck process, the Cox-Ingersoll-Ross model or the Bessel process) are already contained in (19).

We introduce a function $v : \mathbb{R} \rightarrow \mathbb{R}_+$. Furthermore, given S and $v(\cdot)$, we introduce the r.v. $Z_t^T(S, h, v(\cdot))$ defined by

$$Z_t^T(S, h, v(\cdot)) = h \int_t^T v(S_\tau) dB_\tau - \frac{1}{2} h^2 \int_t^T v(S_\tau)^2 d\tau, \quad h \leq 0. \quad (22)$$

We assume the function $v(\cdot)$ to be sufficiently well-behaved for the existence of the integrals on the right-hand side of (22). It is well-known that

$$\mathbb{E}_t \left[e^{Z_t^T(S, h, v(\cdot))} \right] = 1. \quad (23)$$

Recall Definition 3.3. Notice that the r.v. $e^{Z_t^T(S, h, v(\cdot))}$ can be regarded as the continuous-time analog of the Radon-Nikodym derivative used on the right-hand side of (18). Hence, the r.v. $e^{Z_t^T(S, h, v(\cdot))}$ can be used to establish the continuous-time analog of the discrete-time Esscher-Girsanov transform.

Consider a financial derivative security defined by the payoff $g(S_T)$ at time T , for some continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$. Notice that in case S is a traded asset, we remain in a *complete* financial market setting, whereas if S is a non-traded asset the financial market is *incomplete*; see e.g., Duffie (1996), p. 113, for a definition of a *complete* financial market. We introduce a function $\varphi^{(h)} : \mathbb{R} \times [0, T) \rightarrow \mathbb{R}$, with $\varphi^{(h)} \in C^{2,1}(\mathbb{R} \times [0, T))$. Let $\varphi^{(h)}(\cdot, \cdot)$ satisfy the boundary condition $\varphi^{(h)}(S_T, T) = g(S_T)$.

At time $t \in [0, T]$, the price $\pi_t^{v(\cdot)}[g(S_T)]$ of the derivative security $g(S_T)$ based on the Esscher-Girsanov transform, including the time-discount factor, is then given by

$$\pi_t^{v(\cdot)}[g(S_T)] = \int_{[-\infty, 0]} \mathbb{E}_t \left[e^{-\int_t^T r(S_\tau) d\tau} \varphi^{(h)}(S_T, T) e^{Z_t^T(S, h, v(\cdot))} \right] dH(h), \quad (24)$$

for some non-decreasing function $H : [-\infty, 0] \rightarrow [0, 1]$.

Remark 4.1 *The right-hand side of expression (24) can be regarded as a Feynman-Kac path integral; see Feynman & Hibbs (1965). \square*

Then we state the following theorem:

Theorem 4.1 *The Esscher-Girsanov price of the derivative security $g(S_T)$, given in (24), satisfies the martingale property*

$$\pi_t^{v(\cdot)}[g(S_T)] = \int_{[-\infty, 0]} \varphi^{(h)}(S_t, t) dH(h), \quad (25)$$

whenever $\varphi^{(h)}(x, \tau)$ is the solution to the partial differential equation (PDE)

$$\frac{\partial \varphi^{(h)}(x, \tau)}{\partial \tau} + (\mu(x) + hv(x)\sigma(x)) \frac{\partial \varphi^{(h)}(x, \tau)}{\partial x} + \frac{1}{2}\sigma(x)^2 \frac{\partial^2 \varphi^{(h)}(x, \tau)}{\partial x^2} = r(x)\varphi^{(h)}(x, \tau),$$

$$\tau \in (t, T]. \quad (26)$$

Proof: The PDE in (26) coincides with the well-known Kolmogorov Backward Equation (see e.g., Duffie (1996), p. 294) of $\varphi^{(h)}(x, \tau)$, with the drift function adjusted for the change of probability measure as established by the Esscher-Girsanov transform. This proves the stated result. \square

Remark 4.2 Notice that depending on the mixture function $H(\cdot)$ and the function $v(\cdot)$, the price mechanism in (24) can generate an infinite number of prices. \square

In *approximate-arbitrage-free* financial markets (see Duffie (1996), p. 121, for a definition) there exists a probability measure that is equivalent to the “real” probability measure and under which all discounted price processes are martingales. For the original work on the relation between the condition of no arbitrage and the existence of an equivalent martingale measure, we refer to Harrison & Kreps (1979) and Harrison & Pliska (1981). The basic idea of valuation by adjusting the primary asset price process is from Cox & Ross (1976). If the financial market is complete, in addition to being approximate-arbitrage-free, the equivalent martingale measure under which all discounted price processes are martingales is unique, and is found in Theorem 4.2 stated below. If the financial market is incomplete, which is the usual case for (securitized) insurance risks, the derivative price processes cannot be hedged and no arbitrage arguments do in general not supply a unique equivalent martingale measure. In that case, as can be seen from Theorem 4.1, the Esscher-Girsanov transform is a tool to establish a particular (axiomatically justified) equivalent martingale measure.

Theorem 4.2 Suppose that S is a traded asset. If $v(x) = \frac{\mu(x) - r(x)x}{\sigma(x)}$ and

$$H(h) = \begin{cases} 1, & h \geq -1 \\ 0, & \text{otherwise,} \end{cases}$$

then $\pi_t^{v(\cdot)}[g(S_T)]$ coincides with the approximate-arbitrage-free price of the financial derivative $g(S_T)$ at time $t \in [0, T]$.

Proof: The well-known PDE characterization of approximate-arbitrage-free financial derivative prices in a complete financial market (see e.g., Duffie (1996), p. 90) coincides with

the PDE in (26) in case $v(x) = \frac{\mu(x)-r(x)x}{\sigma(x)}$ and $h = -1$. This proves the stated result. \square

Remark 4.3 *The mixture function $H(\cdot)$ derived in Theorem 4.2 can be regarded as a cdf corresponding to a r.v. degenerated at -1 . From an economic point of view, this mixture function corresponds to a representative agent with Arrow-Pratt measure of absolute risk aversion equal to 1. Notice however that if in the economy considered there exists a representative agent with a negative exponential utility function and Arrow-Pratt measure of absolute risk aversion equal to $-h$, the function $v(\cdot)$ derived in Theorem 4.2 can be scaled accordingly.* \square

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