

# Component VaR for a Non-Normal World

Brian G. Peterson <sup>1</sup>      Kris Boudt <sup>2</sup>

---

## Abstract

Value at Risk (VaR) is the most widely used downside risk measure in finance. The contribution to the total portfolio VaR by each component in a portfolio is readily computed under the assumption of normality (Garman, 1997). When assets have non-normal returns, the component risk contributions have previously been much more difficult to calculate. The Cornish-Fisher expansion has recently become popular for estimating the univariate VaR of instruments with non-normal returns because of its accuracy and computational efficiency. We advocate that this estimator, called Modified VaR, is also useful for the estimation of portfolio VaR, because it not only accounts for the skewness and excess kurtosis in the marginal return distributions, but also for the non-linearity in the return dependence structure. We show how to decompose Modified VaR into the component risk of each asset in the portfolio in a manner that is computationally efficient even for very large portfolios. We demonstrate this approach on a portfolio of hedge fund managers to show the utility in a portfolio of assets with non-normal returns.

Keywords: Component Value at Risk, Cornish-Fisher Expansion, Downside Risk, Risk Contribution

---

## Introduction

Value at Risk (VaR) is the most widely used downside risk measure in finance. Garman (1997) in “Taking VaR to Pieces” in this magazine introduced the concept of “Component VaR” and shows that for portfolio VaR computed under the assumption of normality, it is possible to decompose the portfolio risk into the risks introduced by each component of the portfolio. The finance literature on portfolio downside risk has recently realized that it is desirable for estimators of VaR to be decomposed in a financially meaningful way into the risk contributions made by the portfolio holdings (Qian, 2006). It is most common to utilize

positions (with their weights), but any reasonable portfolio decomposition is possible (trades, sectors, instrument types, traders, sub-portfolios) as long as these can be turned into component weights. Such a decomposition of the estimated portfolio VaR is useful for risk analysis, but also for portfolio construction, where a number of analytical methods may be applied to build portfolios using constraints on these measures.

Martin et al. (2001) extended the Component VaR framework to credit portfolios utilizing the saddle-point method, and points the way, along with Gouriéroux et al. (2000), to a general purpose analytical definition of Component VaR. Zangari (1996) extended the RiskMetrics approach to parametric VaR to use the Cornish and Fisher (1937) expansion to correct for skewness and fat tails in the returns. This VaR estimator, called Modified VaR, has become popular because of its high accuracy and computational efficiency. In Boudt et al. (2008), we have applied a similar methodology to create a Modified Expected Shortfall. We have found that component risk decomposition based on the

---

<sup>1</sup> Diamond Management & Technology Consultants, Chicago. E-mail: brian@braverock.com

<sup>2</sup> Faculty of Business and Economics, K.U.Leuven, Belgium. E-mail: Kris.Boudt@econ.kuleuven.be. Financial support from the FWO and K.U.Leuven is gratefully acknowledged.

We thank Peter Carl, Christophe Croux and two anonymous referees for helpful comments and suggestions.

Cornish-Fisher expansion is a better out-of-sample predictor of the magnitude of future component risks than decomposition based on the normal distribution. Because Modified VaR and ES and their components are given by an analytical formula, this methodology is computationally convenient even for portfolios of thousands of assets while accounting for moderate deviations from normality.

This paper first describes the calculation of Modified VaR (MVaR) and its advantages as an estimator of portfolio VaR. We then give the formulas to decompose portfolio MVaR into its components. Finally we conclude with an empirical case in which we provide a detailed description of how this methodology can be implemented and interpreted for a real portfolio of 15 hedge fund managers.

### Modified VaR

Here we present the formulas for calculating VaR for a general portfolio of  $N$  assets with portfolio weights  $w$ . In the remainder of the paper VaR has to be understood as the negative value of a lower quantile (e.g. a 1%-quantile) of the portfolio return, representing the random risk at hand. Denote  $r$  the return on these assets with mean  $\mu$  and covariance matrix  $\Sigma$ . Because these returns can be non-normal, we also need the  $N \times N^2$  co-skewness matrix of the returns

$$M_3 = E[(r - \mu)(r - \mu)' \otimes (r - \mu)']$$

and the  $N \times N^3$  co-kurtosis matrix

$$M_4 = E[(r - \mu)(r - \mu)' \otimes (r - \mu)' \otimes (r - \mu)']$$

where  $\otimes$  stands for the Kronecker product. Denote the  $q$ -th centered portfolio moment  $m_q = E[(r_p - w'\mu)^q]$ . We have that

$$m_2 = w'\Sigma w$$

$$m_3 = w'M_3(w \otimes w)$$

$$m_4 = w'M_4(w \otimes w \otimes w)$$

(see Jondeau and Rockinger, 2006). The portfolio skewness  $s_p$  and excess kurtosis  $k_p$  are then given by

$$s_p = m_3/m_2^{3/2}$$

$$k_p = m_4/m_2^2 - 3.$$

For a loss probability  $\alpha$  such as 1 or 5%, portfolio VaR under the assumption of normally distributed returns is given by

$$\text{GVaR} = -w'\mu - q_\alpha \sqrt{w'\Sigma w} \quad (1)$$

where  $q_\alpha$  is the  $\alpha$ -quantile of the standard normal distribution. GVaR stands for Gaussian VaR.

Since the returns on many financial assets are heavy tailed and skewed, one can obtain a better estimate for VaR by accounting for the deviations from normality. Zangari (1996) provides a Modified VaR calculation that takes the higher moments of non-normal distributions (skewness, kurtosis) into account through the use of a Cornish and Fisher (1937) expansion, better approximating the shape of the true distribution. They define Modified VaR in the following manner:

$$\text{MVaR} = -w'\mu - q_\alpha^{\text{cf}} \sqrt{w'\Sigma w}, \quad (2)$$

where  $q_\alpha^{\text{cf}}$  is the  $\alpha$ -Cornish Fisher quantile. Using the above formulas, this quantile can be easily computed as

$$q_\alpha^{\text{cf}} = q_\alpha + \frac{(q_\alpha^2 - 1)s_p}{6} + \frac{(q_\alpha^3 - 3q_\alpha)k_p}{24} - \frac{(2q_\alpha^3 - 5q_\alpha)s_p^2}{36}. \quad (3)$$

Note that Modified VaR collapses to Gaussian VaR when skewness and excess kurtosis are zero. The Cornish-Fisher expansion encompasses much of the non-normality in returns that could be uncovered by more computationally intensive techniques such as Monte-Carlo simulation or direct fitting of an ideal distribution. This measure is now widely cited and used in the literature, and is usually referred to as ‘‘Modified VaR’’ or ‘‘Cornish-Fisher VaR’’.

Especially for the estimation of portfolio VaR, Modified VaR is more appealing than Gaussian VaR, since in addition to accounting for the asymmetry and heavy tails in the marginal distribution of the component returns, it can account for non-linear dependence between the component returns. Modified VaR uses namely not only correlations, but also higher order cross-moments such as co-skewness and co-kurtosis, to aggregate the individual risks of the portfolio components. To see this, consider a portfolio of two assets with returns  $r_1$  and  $r_2$  and portfolio weights  $w_1$  and  $w_2$ . Without loss of generality, we assume that the returns have zero mean. Denote  $\beta$  the beta (covariance/variance) of asset 1 w.r.t. asset 2. It is always possible to express  $r_2$  as follows

$$r_2 = \beta r_1 + \varepsilon, \quad (4)$$

where the error term  $\varepsilon$  is uncorrelated with  $r_1$ . Under this representation the portfolio return  $r_p$  equals  $(w_1 + w_2\beta)r_1 + w_2\varepsilon$ . From the development of the second, third and fourth moment of  $(w_1 + w_2\beta)r_1 + w_2\varepsilon$ ,

it can be seen that if risk is measured using higher order portfolio moments, then not only correlations but also the higher order cross-moments  $E(r_1^p \varepsilon^q)$  are used for measuring how the risks of the assets join together into portfolio risk.

## Decomposing VaR into Component Risks

We follow the approach presented in Martin et al. (2001) to mathematically decompose portfolio Gaussian and Modified VaR into their component contributions by taking the product of the portfolio weights with the partial derivative of these risk estimators with respect to the weights.<sup>3</sup> Denote by  $w_i$  the weight of asset  $i$  and let  $\partial_i \text{GVaR}$  and  $\partial_i \text{MVaR}$  be the partial derivative of GVaR and MVaR with respect to  $w_i$ . The component of position  $i$  in the portfolio GVaR and MVaR are defined as follows

$$C_i \text{GVaR} = w_i \partial_i \text{GVaR} \quad (5)$$

$$C_i \text{MVaR} = w_i \partial_i \text{MVaR}, \quad (6)$$

where

$$\partial_i \text{GVaR} = -\mu_i - q_\alpha \frac{\partial_i m_2}{2\sqrt{m_2}}$$

$$\partial_i \text{MVaR} = \partial_i \text{GVaR}$$

$$+ \frac{\partial_i m_2}{\sqrt{m_2}} \begin{pmatrix} -(1/12)(q_\alpha^2 - 1)s_p \\ -(1/48)(q_\alpha^3 - 3q_\alpha)k_p \\ +(1/72)(2q_\alpha^3 - 5q_\alpha)s_p^2 \end{pmatrix} \\ + \sqrt{m_2} \begin{pmatrix} -(1/6)(q_\alpha^2 - 1)\partial_i s_p \\ -(1/24)(q_\alpha^3 - 3q_\alpha)\partial_i k_p \\ +(1/18)(2q_\alpha^3 - 5q_\alpha)s_p \partial_i s_p \end{pmatrix}.$$

The derivatives of the portfolio variance, skewness and excess kurtosis can be easily computed by the following formulas

$$\partial_i m_2 = 2(\Sigma w)_i$$

$$\partial_i m_3 = 3(M_3(w \otimes w))_i$$

$$\partial_i m_4 = 4(M_4(w \otimes w \otimes w))_i$$

$$\partial_i s_p = \left( 2m_2^{3/2} \partial_i m_3 - 3m_3 m_2^{1/2} \partial_i m_2 \right) / 2m_2^3$$

$$\partial_i k_p = (m_2 \partial_i m_4 - 2m_4 \partial_i m_2) / m_2^3.$$

<sup>3</sup> We assume that the partial derivative of VaR exists, which implies that the return distribution needs to be sufficiently continuous (Gouriéroux et al., 2000).

Because the portfolio GVaR and MVaR are all 1-homogenous functions of  $w$ , the sum of these risk contributions equals the risk estimate for the overall portfolio. Gouriéroux et al. (2000) and Qian (2006) show that for VaR, this mathematical decomposition of portfolio risk has a financial meaning, establishing that the VaR contribution of each asset equals the asset's expected contribution to the portfolio loss when the portfolio loss equals the negative VaR. Furthermore Tasche (2004) shows that any definition of risk contribution other than partial derivatives are misleading in the context of portfolio optimization.<sup>4</sup>

Of course, one could define VaR components in an analogous way for many other VaR estimators, but for Gaussian and Modified VaR this is particularly appealing because these derivatives are straightforward if tedious to calculate. For many estimation methods, the computation of the derivative of the estimated VaR is challenging or even impossible because the estimator cannot be expressed as an explicit function of the portfolio weights. One exception is the decomposition of Historical VaR (the negative value of the empirical  $\alpha$ -quantile) proposed in Epperlein and Smillie (2006), where the contribution of each component equals

$$C_i \text{HVaR} = \text{HVaR} \frac{\sum_{t=1}^T K_h(-\text{HVaR} - r_{p,t}) w_i r_{t,i}}{\sum_{t=1}^T K_h(-\text{HVaR} - r_{p,t}) r_{p,t}}, \quad (7)$$

where  $r_{t,i}$  ( $r_{p,t}$ ) denotes the return on asset  $i$  (the portfolio) at time  $t$  and  $K_h(\cdot)$  is the triangle kernel with bandwidth  $h$ .<sup>5</sup>

## Application to portfolios of hedge funds

To illustrate the calculation and interpretation of portfolio modified component VaR, we consider the equal-weighted (EW) portfolio for a group of 15 long-running hedge fund managers in the following styles: convertible arbitrage (CA), equity market neutral (EMN), fixed-income arbitrage (FIA), global macro (GM), long short equity (LSE) and merger arbitrage (MA). The EW portfolio can be considered symbolically representative of actual practice in which hedge fund investors pursue style

<sup>4</sup> His argument is linked to Kalkbrenner (2005)'s result that for subadditive and positively homogeneous risk measures, only derivatives yield risk contributions that do not exceed the corresponding stand-alone risks.

<sup>5</sup> Following Epperlein and Smillie (2006) we take  $h$  as the bandwidth that minimizes the mean square error.

Table 1

Sample mean, std dev, skewness, excess kurtosis and the % contribution to 99% GVAR, MVAR and HVAR for the filtered monthly returns of 15 hedge fund managers.

	Mean	Std dev	Skew	exkur	GVAR	MVAR	HVAR
Portfolio	0.009	0.017	0.022	3.512	0.03	0.043	0.041
Style	% Contribution						
1 CA	0.006	0.016	-2.04	11.767	0.032	0.101	0.085
2 CA	0.008	0.033	-0.758	3.545	0.052	0.121	0.093
3 EMN	0.008	0.027	0.338	1.623	0.029	0.049	0.044
4 EMN	0.007	0.039	0.169	2.059	0.003	0.025	0.027
5 FIA	0.008	0.024	0.217	5.438	0.005	0.012	0.015
6 FIA	0.007	0.021	-2.409	13.259	0.005	-0.001	0.008
7 FIA	0.01	0.06	-1.451	8.499	0.035	0.018	0.02
8 GM	0.007	0.038	0.421	0.558	0.063	0.018	0.019
9 GM	0.033	0.133	0.491	6.326	0.453	0.509	0.5
10 GM	0.007	0.037	0.125	0.073	0.01	-0.097	-0.076
11 MA	0.006	0.017	-1.415	4.631	0.02	0.05	0.046
12 MA	0.006	0.014	-1.286	5.427	0.022	0.073	0.065
13 MA	0.005	0.013	-1.087	3.246	0.028	0.066	0.057
14 MA	0.006	0.022	-1.265	2.962	0.053	0.09	0.076
15 LSE	0.01	0.074	1.17	4.697	0.192	-0.038	0.021

diversification. Table 1 presents descriptive statistics for each hedge fund manager, calculated for the monthly returns from October 1996-September 2008 (144 observations).

Sample variance, skewness, and kurtosis and their co-moments are very sensitive to data spikes. To obtain a more stable and robust estimate of these sample moments, we calculate them using a filtered data set rather than the crude data set. We propose this robust filtering in Boudt et al. (2008). When the goal is to calculate VAR for a loss probability  $\alpha$  as it is here, it consists of examining and potentially replacing the most outlying returns that exceed our target quantile with their bounded counterpart. The bounded return has the same orientation as the raw return, but its magnitude is limited by the  $\alpha$ -quantile of the robustly estimated distribution. This reduces the sensitivity of our results to data spikes and outliers while preserving the overall “shape” of the distribution.

We focus on 99% VAR, because of its common usage and because of its inclusion in regulatory regimes such as Basel II. The empirical evidence indicates that almost all hedge fund managers have a return distribution that is heavy-tailed. For example, the

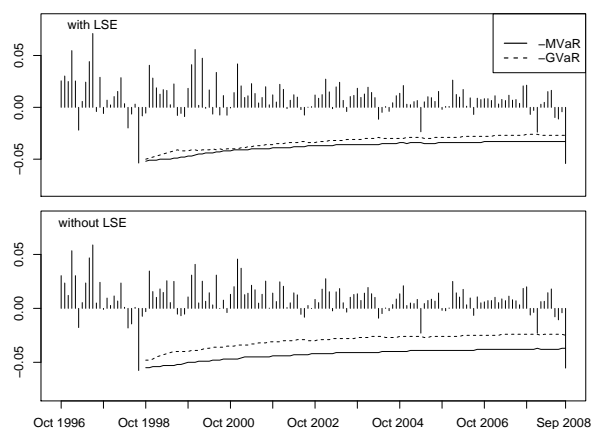
excess kurtosis is very high for the LSE style. This does not mean that investors should avoid this strategy, since LSE offers the attractive combination of the largest positive skewness with a very large excess kurtosis. This intuition is reinforced by comparing portfolios with and without the LSE manager, and provides a good example of the value of component risk decomposition.

The component contributions to GVAR, MVAR and HVAR, as a percentage of the portfolio VAR (with LSE), are reported in the right-hand columns of Table 1. We find that the decomposition based on MVAR more closely resembles the decomposition of HVAR than is the case for GVAR. Because we consider equal-weighted portfolios of 15 assets, the percentage component VARs should be interpreted relative to  $1/15 \approx 0.07$ . If the component risk is above this benchmark value, the asset is a risk intensifier, otherwise it is a risk diversifier. The hedge fund manager pursuing the LSE style is an interesting case. According to the component GVAR measure, it is the second largest contributor to downside risk, while the component MVAR classifies the manager as an important downside risk diversifier. Figures 1 and 2 give more insight into why GVAR and MVAR classify the LSE manager so differently.

Figure 1 plots the monthly returns on the equal-weighted portfolios, with and without the LSE hedge fund manager. We note that the portfolio return in Figure 1 is similar for the current (2008) period and the Russian bond crisis of 1998, where similar credit shocks occurred. The effect of including the LSE style is to induce a positive portfolio skewness and to decrease the excess kurtosis of the portfolio. Instead of a negative skewness of -0.480 it is now 0.022 and excess kurtosis has decreased from 4.061 to 3.512. In Figure 1 we also report the values of Gaussian and Modified VAR computed using the filtered returns up to the previous month. We see that for the portfolio with LSE, the Gaussian and Cornish-Fisher quantiles are similar. For the portfolio without LSE, the negative returns are larger and more numerous, leading to a significantly larger estimate for MVAR.

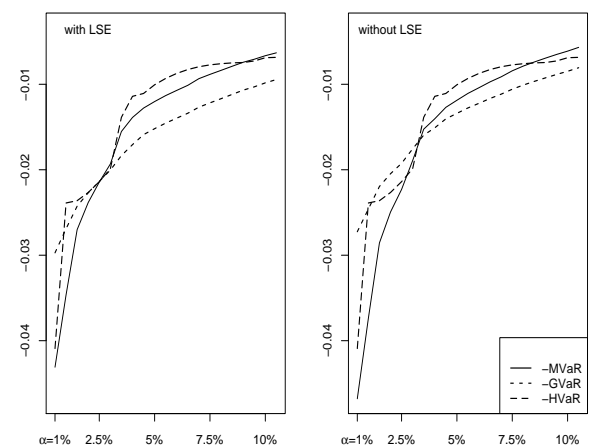
In Figure 2 we compare GVAR, MVAR and HVAR for different values of the loss probability  $\alpha$ . Note that compared with GVAR, MVAR more closely resembles the historical quantile across all confidence levels, visually indicating that the LSE hedge fund manager is a risk diversifier and not, as the component GVAR indicates, a risk intensifier.

Fig. 1. Monthly returns on equal-weighted portfolio with and without the LSE manager



Note: the negative value of GVAR and MVAR is based on the filtered returns from inception up to the previous month

Fig. 2. Negative value of MVAR, GVAR and HVAR of an equal-weighted portfolio with and without LSE, for different values of the loss probability  $\alpha$



## Closing remarks

In this article we have shown how to estimate the Component VaR of a portfolio of assets with non-normal returns by constructing a Component Modified VaR. We advocate that for the estimation of portfolio VaR, Modified VaR is useful, because it not only addresses the asymmetry and heavy tails in the marginal distributions of the portfolio returns, it also accounts for the non-linear dependence between the assets when aggregating their individual risks. We have further illustrated the usefulness of

this technique with portfolios of hedge fund managers.<sup>6</sup>

The computational efficiency of the methods presented here should make incorporating component decomposition feasible at multiple levels within a large organization without major infrastructure changes. Because Component VaR satisfies the property of additivity, VaR decomposition, once calculated at an individual position level, allows the portfolio to be aggregated or disaggregated on multiple different criteria (trading desk, asset class, sector, industry, style) to provide different views into the risk of the entire portfolio. Risk decomposition at multiple levels of granularity should be utilized as part of a comprehensive risk monitoring regime.

In most portfolios of more than a few non-normal assets, assessing the marginal impact of a change to portfolio allocations involves running simulations to establish the possible distribution of losses of the new combined portfolio. While simulation will remain an important tool in a portfolio manager's process, component decomposition analysis should improve the manager's understanding of the impact of a real or proposed position change on the portfolio.

The additional information provided by portfolio risk decomposition techniques adds significant information to the portfolio selection, construction, and management processes. In any portfolio holding a large number of assets, there will be many possible portfolios with similar mean and standard deviation. A practical conclusion from examining the component risk contributions to the sample portfolios would be that the long-term performance of the portfolio could be improved by adjusting the component weights to better match a deliberate risk profile that is complimentary to the investor's goals.

<sup>6</sup> The functions used in this paper are available in the R package *PerformanceAnalytics* (Carl and Peterson, 2008) or directly from the authors.

## References

- Boudt, K., Peterson, B. G., Croux, C., 2008. Estimation and Decomposition of Downside Risk for Portfolios with Non-Normal Returns. *Journal of Risk*, forthcoming.
- Carl, P., Peterson, B. G., 2008. PerformanceAnalytics: Econometric Tools for Performance and Risk Analysis. R package version 0.9.7.  
URL <http://braverock.com/R/>
- Cornish, E. A., Fisher, R. A., 1937. Moments and Cumulants in the Specification of Distributions. *Revue de l'Institut International de Statistique* 5 (4), 307–320.
- Epperlein, E., Smillie, A., August 2006. Cracking VaR with Kernels. *Risk*, 70–74.
- Garman, M. B., 1997. Taking VaR to pieces. *RISK* 10 (10), 70–71.
- Gouriéroux, C., Laurent, J.-P., Scaillet, O., 2000. Sensitivity Analysis of Value at Risk. *Journal of Empirical Finance* 7 (3-4), 225–245.
- Jondeau, E., Rockinger, M., 2006. Optimal Portfolio Allocation Under Higher Moments. *European Financial Management* 12 (1), 29–55.
- Kalkbrener, M., July 2005. An Axiomatic Approach to Capital Allocation. *Mathematical Finance* 15 (3), 425–437.
- Martin, R., Thompson, K., Browne, C., 2001. VaR: Who Contributes and How Much? *RISK* 14 (8), 99–102.
- Qian, E., 2006. On the Financial Interpretation of Risk Contribution: Risk Budgets Do Add Up. *Journal of Investment Management* 4 (4), 1–11.
- Tasche, D., 2004. Economic Capital: A Practitioner Guide. *Risk Books*, Ch. Allocating Portfolio Economic Capital to Sub-Portfolios, pp. 275–302.
- Zangari, P., 1996. A VaR Methodology for Portfolios that include Options. *RiskMetrics Monitor First Quarter*, 4–12.