A note on some new perpetuities

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Abstract

In a recent paper, Salminen and Yor (2004b) relate the distribution of the Dufresne's reflected perpetuity

$$I^+ = \int_0^{+\infty} e^{-2B^\mu(s)} 1_{(B^\mu(s)>0)} ds$$

where B^μ is Brownian motion with drift $\mu > 0$, to the hitting time of a reflected Bessel process. In this contribution, we adapt the results of Salminen and Yor (2004b) in several ways. First, we use spectral theory to obtain a series expansion for the distribution of I^+ that renders this quantity applicable to actuarial purposes. We also study the exponential functionals

$$I^\alpha = \int_0^{+\infty} e^{-2X_\alpha^\mu(s)} ds$$

where X_α^μ is a skew Brownian motion with drift $\mu > 0$.
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1 Introduction

In his seminal paper, Dufresne (1990) proves with somewhat complicated arguments that the perpetuity

$$I = \int_0^{+\infty} e^{-2B^{\mu}(s)} ds \tag{1}$$

where B^{μ} is Brownian motion (BM) with drift $\mu > 0$, is distributed as the reciprocal of a Gamma variable. Since then many authors have been interested in this quantity. We can cite De Schepper, Goovaerts and Delbaen (1992), Yor (1992) and Milevsky (1997) and (1999). The perpetuity (1) arises in many applications in finance and insurance. In life insurance business, we can interpret the random variable I as the initial endowment needed to fund a unit continuous perpetuity, see *e.g.* Milevsky (1997). The process $(2B^{\mu}(t), t \geq 0)$ is then the stochastic rate of return on investments. When $\mu > 0$, $+\infty$ is an *attracting* boundary for $2B^{\mu}$ which ensures the boundness of the variable I.

For $t < +\infty$, the rate of return $2B^{\mu}(t)$ can reach negative values with non-zero probability. In finite time horizon, the rate of return on equity investments may be negative. Nevertheless, in case the initial endowment is invested in fixed income instruments, one expects the model to preclude negative rates of return[1]. A possible answer to this problem is to impose at the origin a reflecting boundary to the process $2B^{\mu}$.

In a recent paper, Salminen and Yor (2004b) obtain the Laplace transform of the Dufresne's reflected perpetuity

$$I^{+} = \int_0^{+\infty} e^{-2B^{\mu}(s)} 1_{(B^{\mu}(s)>0)} ds.$$

Implementing the time change $\tau(t) = \inf \left\{ s : \int_0^s 1_{(B^{\mu}(u)>0)} du > t \right\}$, they prove that the perpetuity I^{+} is equal in law to

$$I^{1} = \int_0^{+\infty} e^{-2X_1^{\mu}(s)} ds \tag{2}$$

where X_1^{μ} is a reflecting BM with drift $\mu > 0$. In this paper, we inverse the Laplace transform and we compare numerically the distribution of the perpetuities I and I^{+}. We also extend the results of Salminen and Yor

[1]We thank Moshe Milevsky for fruitful discussions on this subject

(2004) replacing X_1^μ by a more general skew BM, see *e.g.* Harrison and Shepp (1981). The paper is organized as follows. In section 2, we recall the construction of skew BM and we prove a time change relation useful to relate perpetuities and first hitting times. In section 3, we give a series expansion for the distribution of I^1 using spectral theory and we compute the Laplace of I^α. In section 4, we illustrate numerically the results.

2 Useful time change identities

A skew Brownian motion with parameter $\alpha \in [0,1]$ behaves like a Brownian motion away from the origin and is reflected to the positive side with probability α and to the negative side with probability $1-\alpha$ when it arrives at 0. It is shown, see Harrison and Shepp (1981), that the skew Brownian motion is the semimartingale with decomposition

$$X_\alpha(t) = B(t) + (2\alpha - 1)L_t^0(X_\alpha)$$

where $(B(t), t \geq 0)$ is an adapted Brownian motion and $L_t^0(X_\alpha)$ is the symmetric local time associated to X_α. More generally, we define the skew Brownian motion with drift μ as a solution of the Stochastic Differential Equation (SDE)

$$dX_\alpha^\mu(t) = \mu dt + dB(t) + (2\alpha - 1)dL_t^0(X_\alpha^\mu), \quad t > 0 \tag{3}$$

and $X_\alpha^\mu(0) = x$. We are interested in the distribution of the perpetuities

$$I_g^\alpha = \int_0^{+\infty} g(X_\alpha^\mu(s))ds \tag{4}$$

with $\mu > 0$, with emphasis on the case $g(x) = e^{-2x}$ for financial and actuarial applications. In this respect, we will use the following Proposition which is an elementary generalization of Proposition 2.1. in Salminen and Yor (2004b).

Proposition 1 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a strictly monotone continuous function, f' and f'' exist and are continuous on $\mathbb{R}$ except at 0 where the limits*

$$f'(0\pm) = \lim_{x \to 0\pm} f'(x), \qquad f''(0\pm) = \lim_{x \to 0\pm} f''(x)$$

are finite. Let X_α^μ be a skew Brownian motion with drift μ starting at x, then

$$f\left(X_\alpha^\mu(t)\right) = Z\left(\int_0^t f'^2\left(X_\alpha^\mu(s)\right) ds\right) \tag{5}$$

where Z is a solution of the SDE

$$dZ(t) = d\beta(t) + (G \circ f^{-1})(Z(t))dt + (2\beta - 1)dL_t^{f(0)}(Z), \quad t > 0$$

and $Z(0) = f(x)$, with $G(x) = \left(\frac{1}{2}f''(x) + \mu f'(x)\right) / f'^2(x)$ and

$$\beta = \frac{(1-\alpha)f'(0-)}{\alpha f'(0+) + (1-\alpha)f'(0-)}.$$

Proof. The function f is the difference of two convex functions. Thus, an application of Tanaka formula to the process $Y = f\left(X_\alpha^\mu\right)$ yields

$$dY(t) = f'\left(X_\alpha^\mu(t)\right)dB(t) + f'^2\left(X_\alpha^\mu(s)\right)G\left(X_\alpha^\mu(s)\right)dt + \gamma dL_t^0\left(X_\alpha^\mu\right)$$

where $\gamma = \alpha f'(0+) - (1-\alpha)f'(0-)$. Implementing the random time change $\tau(t) = \inf\{s : \int_0^s f'^2(X_\alpha^\mu(u))du > t\}$, we obtain

$$\begin{aligned} Y(\tau(t)) - Y(0) &= \beta(t) + \int_0^t G\left(X_\alpha^\mu(\tau(s))\right)ds + \gamma L_\tau^0\left(X_\alpha^\mu\right) \\ &= \beta(t) + \int_0^t \left(G \circ f^{-1}\right)\left(Y(\tau(s))\right)ds + \gamma L_t^0\left(X_\alpha^\mu(\tau)\right) \quad (6) \end{aligned}$$

where $\beta(t)$ is a Brownian motion according to the Dambis, Dubins-Schwarz's Theorem. The following relations complete the proof, see e.g. Revuz and Yor (1998),

$$\begin{aligned} L_t^{f(0)}(Y) &= \frac{1}{2}\left(L_t^{f(0+)}(Y) + L_t^{f(0-)}(Y)\right) \\ &= -\frac{f'(0+)}{2}L_t^{0+}\left(X_\alpha^\mu\right) - \frac{f'(0-)}{2}L_t^{0-}\left(X_\alpha^\mu\right) \\ &= -\left(\alpha f'(0+) + (1-\alpha)f'(0-)\right)L_t^0\left(X_\alpha^\mu\right), \quad (7) \end{aligned}$$

since $L_t^{0+}\left(X_\alpha^\mu\right) = 2\alpha L_t^0\left(X_\alpha^\mu\right)$ and $L_t^{0-}\left(X_\alpha^\mu\right) = 2(1-\alpha)L_t^0\left(X_\alpha^\mu\right)$. □

Remark 1 *As the function f is strictly monotone, we can verify that $\beta \in [0, 1]$.*

3 Perpetuities

When the function $g = f'^2$ is subject to additional conditions, the perpetuity

$$I_g = \int_0^{+\infty} g\left(B(s) + \mu s\right)ds \quad (8)$$

is equal in law to the first hitting time of a diffusion process. We refer to Salminen and Yor (2004a) and (2004b) for a complete account. Using Proposition 1, we can use the same approach to compute the distribution of perpetuities involving skew Brownian motion and discount functions g discontinous at one point. If the limit $\lim_{x\to+\infty} f(x) = r$ exists, the perpetuity

$$I_{f'^2} = \int_0^{+\infty} f'^2\left(X_\alpha^\mu(s)\right) ds$$

with $\mu > 0$, is equal in law to the first hitting time $H_r(Z) = \inf\{t \geq 0 : Z(t) = r\}$ of the process Z defined in Proposition 1. Indeed, $+\infty$ is an attracting boundary for the skew Brownian motion with drift $\mu > 0$ and $\alpha \in (0, 1]$, thus

$$\begin{aligned} \lim_{t\to+\infty} f\left(X_\alpha^\mu(t)\right) &= Z\left(I_{f'^2}\right) \\ &= r. \end{aligned}$$

In case f is assumed to be decreasing, we conclude that $f\left(X_\alpha^\mu(t)\right) > r$ and $I_{f'^2} = H_r(Z)$.

The process Z is a linear diffusion (a strong Markov process with continuous paths). The associated scale function $s(x) = \int^x s'(z)dz$ and speed density $m(x)$ give rise to the following representation of the infinitesimal generator

$$\mathcal{G}u(x) = \frac{d}{dm}\frac{d}{ds}u(x). \tag{9}$$

The following Lemma provides the speed and scale densities of the process Z defined in Proposition 1. We assume that $\beta \in (0, 1)$ to avoid trivial complications.

Lemma 1 *Let Z be a solution of the SDE*

$$dZ(t) = d\beta(t) + (G \circ f^{-1})(Z(t))dt + (2\beta - 1)dL_t^{f(0)}(Z)$$

with $G(x) = \left(\frac{1}{2}f''(x) + \mu f'(x)\right)/f'^2(x)$ and $\beta \in (0, 1)$, then

$$\begin{aligned} s'(x) &= e^{-\int^x 2(G\circ f^{-1})(z)dz}/r'_\beta(x) \\ m(x) &= 2/s'(x), \end{aligned}$$

where $r'_\beta(x) = \frac{1}{1-\beta}1_{(x\geq f(0))} + \frac{1}{\beta}1_{(x<f(0))}$.

Proof. Consider the process $(Y(t), t \geq 0)$ solution of the following SDE

$$dY(t) = \frac{(G \circ f^{-1} \circ r_\beta)(Y(t))}{r'_\beta(Y(t))}dt + \frac{1}{r'_\beta(Y(t))}dB(t)$$

where $r_\beta(x) = \int^x r'_\beta(z)dz$. The process Y is a linear diffusion with scale and speed densities given by

$$\begin{aligned} s'_Y(y) &= e^{-\int^{r_\beta(y)} 2(G\circ f^{-1})(z)dz} \\ m_Y(y) &= 2/s'_Y(y). \end{aligned}$$

We can verify using Tanaka formula that $Z = r_\beta(Y)$. The scale and speed densities of Z result from the relations $s'(x) = s'_Y(r_\beta^{-1}(x))/r'_\beta(r_\beta^{-1}(x))$ and $m(x) = m_Y(r_\beta^{-1}(x))/r'_\beta(r_\beta^{-1}(x))$. □

3.1 Dufresne's reflected perpetuities

The Dufresne's reflected perpetuity corresponds to $f(x) = e^{-x}$ and $\alpha = 1$. Salminen and Yor (2004) relate this perpetuity to the hitting time at the origin of a reflected Bessel process Z and provide the Laplace transform of its distribution. In this paper, we inverse the Laplace transform using the spectral representation of the transition density of Z. An application of Proposition 1 yields

$$(G \circ f^{-1})(x) = \frac{1 - 2\mu}{2x}$$

and $\beta = 0$. We conclude that the Dufresne's reflected perpetuity is equal in law to the first hitting at 0 of the process solution of the reflected SDE

$$dZ(t) = dB(t) + \frac{1 - 2\mu}{2Z(t)}dt - dL_t^1(Z), \quad t > 0 \tag{10}$$

and $Z(0) = e^{-x}$. The process Z instantaneously killed at 0 is a linear diffusion with infinitesimal generator

$$\mathcal{G}u(x) = \frac{1}{2}u''(x) + \frac{1 - 2\mu}{2x}u'(x)$$

acting on the domain $\mathcal{D} = \left\{u : u, \mathcal{G}u \in C_b((0, 1]), u(0+) = 0, \frac{du}{ds}(1-) = 0\right\}$. The diffusion Z takes values in the bounded interval $(0, 1]$ and the spectral representation of its transition density *w.r.t.* Lebesgue measure reduces to the series expansion given is the next proposition, see *e.g.* Linetsky (2004).

Proposition 2 *Let $F(t) = Prob\,[I^+ \leq t]$ be the distribution function of the perpetuity*

$$I^+ = \int_0^{+\infty} e^{-2X_1^\mu(s)} ds$$

where X_1^μ is a BM with drift $\mu > 0$ reflected at 0 and starting at $x \geq 0$. Then, $F(t) = 1 - \int_0^1 p(t, e^{-x}, y)dy$ with the following spectral expansion for $p(t, x, y)$

$$p(t, x, y) = y^{-2\mu+1} \sum_{n=1}^{+\infty} e^{-\lambda_n t} \varphi_n(x)\varphi_n(y)\pi_n,$$

where $\pi_n^{-1} = \int_0^a \varphi_n^2(x)x^{-2\mu+1}dx$. The eigenfunction $\varphi_n(x)$ is $x^\mu\ J_\mu(x\sqrt{2\lambda_n})$ where $J_\mu(x)$ is the first kind Bessel function of (fractional) order μ and the eigenvalues $(\lambda_n, n \in \mathbb{N}_0)$ are the strictly positive zeros of

$$\sqrt{2\lambda} J_\mu'(\sqrt{2\lambda}) + \mu J_\mu(\sqrt{2\lambda}).$$

Proof. The perpetuity I^+ is equal in law to the first hitting time at the origin of the process Z starting at e^{-x}. The probability that $H_0(Z)$ is strictly greater than t is given by

$$\int_0^1 p(t, e^{-x}, y)dy$$

where $p(t, x, y)$ is the transition density of Z killed at his first visit at the origin.

The speed density of the process Z instantaneously killed at 0 is equal to $m(y) = y^{-2\mu+1}$ and the scale function is $s(x) = x^{2\mu}$, see Borodin and Salminen (1996). As the domain of Z is bounded, the spectrum of the operator $\mathcal{G}$ is a countable set $(\lambda_n, n \in \mathbb{N})$ and $p(t, x, y)$ reduces to the series

$$p(t, x, y) = m(y) \sum_{n=0}^{+\infty} e^{-\lambda_n t} \varphi_{\lambda_n}(x)\varphi_{\lambda_n}(y) \tag{11}$$

up to some normalizing factor. The eigenfunction $\varphi_\lambda(x)$ is the continuous solution with continuous scale derivative of the ODE

$$\frac{1}{2}u''(x) + \frac{1-2\mu}{2x}u'(x) = -\lambda u(x) \tag{12}$$

such that $\varphi_\lambda(x)$ is m-square integrable and satisfies the boundary conditions $\varphi_\lambda(0) = 0$ and $\frac{d\varphi_\lambda}{ds}(1-) = 0$. The functions $x^\mu\ J_\mu(x\sqrt{2\lambda})$ and

$x^\mu\ Y_{(-\mu)}(x\sqrt{2\lambda})$ where $J_\mu(x)$ and $Y_{(-\mu)}(x)$ are respectively Bessel functions of first and second kind, span the solutions of the ODE (12). Hence, the eigenfunctions are in the form $u(x) = c_1\ x^\mu\ J_\mu(x\sqrt{2\lambda}) + c_2\ x^\mu\ Y_{(-\mu)}(x\sqrt{2\lambda})$. With the condition $u(0) = 0$, we find that $c_2 = 0$. The eigenvalues follow from the condition $u'(1-) = 0$. □

Remark 2 *For a general volatility $\sigma > 0$, the perpetuity*

$$I_\sigma^+ = \int_0^{+\infty} e^{-2Y(s)} ds$$

where $Y(t) = \sigma B(t) + \mu t + L_t^0(Y)$, is equal in law to $\int_0^{+\infty} e^{-2X_1^{\mu/\sigma^2}(s)} ds/\sigma^2$. This can be proved using the time change $t' = \sigma^2 t$ and the scaling property of BM and local times, see e.g. Revuz and Yor (1998).

3.2 Skew perpetuities

In this section, we derive the Lalpace tansform of the perpetuity

$$I^\alpha = \int_0^{+\infty} e^{-2X_\alpha^\mu(s)} ds \tag{13}$$

where $X_\alpha^\mu(t) = x + B(t) + \mu t + (2\alpha - 1)L_t^0(X_\alpha^\mu)$ is a skew Brownian motion with drift $\mu > 0$ and skew parameter $\alpha \in (0, 1)$. An application of Proposition 1 yields

$$(G \circ f^{-1})(x) = \frac{1 - 2\mu}{2x}$$

and $\beta = 1 - \alpha$. We conclude that the skew perpetuity (13) is equal in law to the first hitting at 0 of the process solution of the SDE involving its local time

$$dZ(t) = dB(t) + \frac{1 - 2\mu}{2Z(t)} dt + (1 - 2\alpha) dL_t^1(Z), \quad t > 0 \tag{14}$$

and $Z(0) = e^{-x}$. The process Z instantaneously killed at 0 is a linear diffusion with infinitesimal generator

$$\mathcal{G}u(x) = \frac{1}{2}u''(x) + \frac{1 - 2\mu}{2x} u'(x)$$

acting on the domain $\mathcal{D} = \{u : u, \mathcal{G}u \in C_b((0, +\infty)), u(0+) = 0, \alpha u'(1-) = (1 - \alpha)u'(1+)\}$. The following Proposition provides the Laplace transform of I^α.

Proposition 3 *The Laplace transform of I^α is given by*

$$E_{e^{-x}}\left[e^{-sI^\alpha}\right] = \frac{\psi_s(e^{-x})}{\psi_s(0+)}$$

where ψ_s is the decreasing fundamental solution of $\mathcal{G}u = su$,

$$\psi_s(x) = \begin{cases} C_1 x^\mu I_\mu(\sqrt{2s}x) + C_2 x^\mu K_\mu(\sqrt{2s}x), & x \le 1 \\ x^\mu K_\mu(\sqrt{2s}x), & x > 1 \end{cases} \tag{15}$$

where

$$\begin{aligned} C_1 &= \frac{2\alpha-1}{\alpha}\sqrt{2s}K_\mu(\sqrt{2s})K_{\mu-1}(\sqrt{2s}) \\ C_2 &= 1 - \frac{2\alpha-1}{\alpha}\sqrt{2s}I_\mu(\sqrt{2s})K_{\mu-1}(\sqrt{2s}). \end{aligned}$$

Proof. The perpetuity I^α is equal in law to the first hitting time at the origin of the process Z starting at e^{-x}. The Laplace transform of $H_0(Z)$ is given by

$$E_{e^{-x}}\left[e^{-sH_0(Z)}\right] = \frac{\psi_s(e^{-x})}{\psi_s(0+)}$$

where ψ_s is the decreasing fundamental solution of $\mathcal{G}u = su$, see Borodin and Salminen (1996). The continuous function ψ_s satisfies the conditions

$$\psi_s(0+) < +\infty, \quad \alpha\psi_s'(1-) = (1-\alpha)\psi_s'(1+), \quad \psi_s(+\infty) = 0.$$

We observe that the functions $x^\mu I_\mu\left(x\sqrt{2s}\right)$ and $x^\mu K_\mu\left(x\sqrt{2s}\right)$ span the solutions of $\mathcal{G}u = su$ and we seek a solution in the form

$$\psi_s(x) = \begin{cases} C_1 x^\mu I_\mu(\sqrt{2s}x) + C_2 x^\mu K_\mu(\sqrt{2s}x), & x \le 1 \\ C_3 x^\mu I_\mu(\sqrt{2s}x) + C_4 x^\mu K_\mu(\sqrt{2s}x), & x > 1. \end{cases}$$

As $\psi_s(+\infty) = 0$, we conclude that $C_3 = 0$. The conditions $\psi_s(1+) = \psi_s(1-)$ and $\alpha\psi_s'(1-) = (1-\alpha)\psi_s'(1+)$ lead to the constants C_1 and C_2[2] (to some normalizing factor). □

[2] Use is made of the following equalities of Bessel functions, see *e.g.* Gradshteyn and Ryzik (1980),

$$\begin{aligned} I_\nu(z)K_{\nu+1}(z) + K_\nu(z)I_{\nu+1}(z) &= 1/z \\ \frac{d}{dz}\left(z^\nu I_\nu(z)\right) &= z^\nu I_{\nu-1}(z) \\ \frac{d}{dz}\left(z^\nu K_\nu(z)\right) &= -z^\nu K_{\nu-1}(z). \end{aligned}$$

Remark 3 *For completeness, we mention that the Laplace transform of the perpetuity*

$$I = \alpha \int_0^{+\infty} e^{-2B^\mu(s)} 1_{(B^\mu(s)<0)} ds + \beta \int_0^{+\infty} e^{-2B^\mu(s)} 1_{(B^\mu(s)\geq 0)} ds$$

can be obtained with similar arguments. This quantity is investigated in Salminen and Yor (2003) using excursion theory, see e.g. Yor (1995)[3].

Remark 4 *Similar results were obtained with somewhat different arguments in Urbanik (1992)*[4].

4 Numerical illustrations

In this section, we compare the distribution of the initial endowment needed to fund a perpetuity when the rate of return can reach negative values or not. We model the rate of return on investments consecutively by a BM with drift 0.08 ($\mu = 0.04$) and volatility 0.15 ($\sigma = 0.19$) and the same process with a reflecting boundary at the origin. Figure 2 compares the distribution of I and I^+ with $k = 15$ terms in the series expansion given in Proposition 2. As we expect, the perpertuiy I has heavier right tail than I^+.

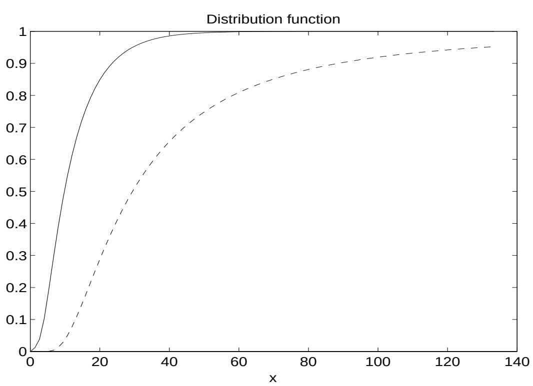

Figure 1: Distribution function of Dufresne's perpetuity (in dashed line) and reflected Dufresne's perpetuity (in solid line) with $k = 15$ terms in the series.

[3]We thank Marc Yor for bringing to our attention these results

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