A Note on Additive Risk Measures in Rank-Dependent Utility

Marc J. Goovaerts
Catholic University of Leuven
and University of Amsterdam

Rob Kaas
University of Amsterdam

Roger J.A. Laeven[*]
Tilburg University
and CentER

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Abstract

This note proves that risk measures obtained by applying the equivalent utility principle in rank-dependent utility are additive if and only if the utility function is linear or exponential and the probability weighting (distortion) function is the identity.

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[*]Corresponding author. E-mail: R.J.A.Laeven@uvt.nl, Phone: +31 13 466 2430, Fax: +31 13 466 3280.

1 Introduction

In this note, we consider rank-dependent utility of a risk. Rank-dependent utility was first proposed by Quiggin (1982) under the guise of anticipated utility; see also Yaari (1987). It coincides with Choquet expected utility of Schmeidler (1989) for *decision-making under risk*, that is, decision-making in settings where the probability measure is objective, and known and given in advance, which is the setting considered here. It further coincides with (and was in fact at the basis of) cumulative prospect theory of Tversky and Kahneman (1992) for decision-making under risk, when restricting to either positive or negative outcomes.

From this rank-dependent utility, we derive a risk measure $\rho[X]$ as that displacement of the risk X that makes the rank-dependent utility $\pi\big[X - \rho[X]\big]$ of the shifted risk equal to zero, see Denuit *et al.* (2006) and Goovaerts *et al.* (2010). Throughout, realizations of X designate monetary payoffs.

We show that if the rank-dependent utility $\pi[X]$ involves a concave utility function next to its probability weighting (or, distortion) function and moreover the risk measure $\rho[X]$ is *additive* for independent risks, then the probability weighting function must be linear and the utility function must be linear or exponential. Thus, we generalize the following results. Gerber (1974a, 1985) proved that in the absence of a distortion function (that is, if it is the identity), the utility function must be linear or exponential. Heilpern (2003) proves that with linear or exponential utility, the distortion function must be linear to make ρ additive. Our proof proceeds by using the additivity property for simple Bernoulli risks to derive a differential equation for the utility function u, and proving that its MacLaurin expansion has the required coefficients.

Additivity of risk measures used for premium calculation was advocated already by Borch (1962, p. 429): "It is natural to require that the company shall receive the same amount whether it accepts the two [independent] portfolios separately or in one single transaction."

2 The Result

Our main result is the following.

Theorem 2.1 *Consider a utility function u and a distortion function g with*

(i) $u : \mathbb{R} \to \mathbb{R}$ is a function with a MacLaurin expansion, normalized in the sense that $u(0) = 0$, $u'(0) = 1$, strictly increasing and concave, so $u' > 0$, $u'' \leq 0$;

(ii) $g : [0,1] \to [0,1]$ *has* $g(0) = 0$, $g(1) = 1$, $g' \geq 0$, g'' *exists and* $g'(1) \neq 0$.

For any random variable X*, consider its rank-dependent utility*

$$\pi[X] := \int_{-\infty}^{0} (g(\mathbb{P}[X > x]) - 1) du(x) + \int_{0}^{+\infty} g(\mathbb{P}[X > x]) du(x).$$

Now define the risk measure ρ *as the solution to the equation* $\pi[X - \rho] = 0$*, that is, by the principle of equivalent utility. Then we get an* additive *risk measure if and only if* g *is the identity function and* u *is exponential or linear.*

Proof: Notice first that (i) –in particular, the assumptions that u has a MacLaurin expansion and is strictly increasing with $u'(0) = 1$– and (ii) –in particular, the assumptions that $g(0) = 0$, $g(1) = 1$ and $g' \geq 0$– guarantee that the equation $\pi[X - \rho] = 0$ admits at most one solution. If, for given X, u, g, the equation $\pi[X - \rho] = 0$ does not have a solution, X is uninsurable under rank-dependent utility with utility function u and distortion function g.

The 'if'-part of the theorem is trivial. Let us prove the 'only if'-part. Let $X_{c,t}$ be a Bernoulli risk with parameters c, t such that $\mathbb{P}[X_{c,t} = 0] = t$ and $\mathbb{P}[X_{c,t} = c] = 1 - t$, so

$$\mathbb{P}[X_{c,t} - \rho > x] = \mathbb{P}[X_{c,t} > x + \rho] = \begin{cases} 1 & \text{for } x < -\rho, \\ 1 - t & \text{for } -\rho \leq x < c - \rho, \\ 0 & \text{for } c - \rho \leq x. \end{cases}$$

For such a risk, we get, using $\rho \leq c$,

$$\begin{aligned} \pi[X_{c,t} - \rho] &= -\int_{-\rho}^{0} (1 - g(1-t)) du(x) + \int_{0}^{c-\rho} g(1-t) du(x) \\ &= (1 - g(1-t)) u(-\rho) + g(1-t) u(c - \rho) \\ &= 0. \end{aligned}$$

Upon differentiation with respect to t (with c fixed) we find

$$\begin{aligned} &g'(1-t)u(-\rho) + (1 - g(1-t))u'(-\rho) \left(-\frac{d\rho}{dt} \right) \\ &- g'(1-t)u(c-\rho) + g(1-t)u'(c-\rho) \left(-\frac{d\rho}{dt} \right) = 0. \end{aligned}$$

Now we set $t = 0$, so $\rho = c$, and the first term becomes $g'(1)u(-c)$. By our assumptions (i) and (ii), the second and third term vanish, and the fourth term reduces to $-\rho'(0)$. So we have

$$g'(1)u(-c) = \rho'(0). \tag{1}$$

We excluded $g'(1) = 0$ in (ii). Note that if this should hold, so $\rho'(0) = 0$, the risk measure ρ would be insensitive to losing an amount c with probability 1, which is unnatural. Differentiating once more with respect to t yields

$$\begin{aligned}
&- g''(1-t)u(-\rho) - 2g'(1-t)u'(-\rho)\frac{d\rho}{dt} \\
&+ (1 - g(1-t))\left(u''(-\rho)\left(\frac{d\rho}{dt}\right)^2 - u'(-\rho)\left(\frac{d^2\rho}{dt^2}\right)\right) \\
&+ g''(1-t)u(c-\rho) + 2g'(1-t)u'(c-\rho)\frac{d\rho}{dt} \\
&+ g(1-t)\left(u''(c-\rho)\left(\frac{d\rho}{dt}\right)^2 - u'(c-\rho)\frac{d^2\rho}{dt^2}\right) = 0,
\end{aligned}$$

which, by our assumptions, for $t = 0$ reduces to

$$-g''(1)u(-c) - 2g'(1)u'(-c)\rho'(0) + 2g'(1)\rho'(0) + u''(0)\left(\rho'(0)\right)^2 = \rho''(0). \tag{2}$$

Now let $S := X_{c,t} + Y_{c,t}$, where $X_{c,t}$ and $Y_{c,t}$ are two independent and identically distributed Bernoulli risks, so,

$$\mathbb{P}[S - 2\rho > x] = \mathbb{P}[S > x + 2\rho] = \begin{cases} 1 & \text{for } x < -2\rho; \\ 1 - t^2 & \text{for } -2\rho \le x < c - 2\rho; \\ (1-t)^2 & \text{for } c - 2\rho \le x < 2(c-\rho); \\ 0 & \text{for } 2(c-\rho) \le x. \end{cases}$$

Then, provided that $c \le 2\rho \le 2c$,

$$\begin{aligned}
\pi[S - 2\rho] &= -\int_{-2\rho}^{c-2\rho} (1 - g(1-t^2))du(x) - \int_{c-2\rho}^{0} (1 - g((1-t)^2))du(x) \\
&\quad + \int_0^{2(c-\rho)} g((1-t)^2)du(x) \\
&= (1 - g(1-t^2))u(-2\rho) + (g(1-t^2) - g((1-t)^2))u(c-2\rho) \\
&\quad + g((1-t)^2)u(2(c-\rho)) \\
&= 0.
\end{aligned}$$

The first derivative with respect to t at $t = 0$ (where $c \le 2\rho \le 2c$ is satisfied), by the assumed additivity of ρ, again leads to equation (1):

$$g'(1)u(-c) = \rho'(0).$$

The second derivative at $t = 0$, again assuming additivity of ρ, yields

$$2g'(1)u(-2c) - 4(g'(1) + g''(1))u(-c) - 8g'(1)u'(-c)\rho'(0) \\ +8g'(1)\rho'(0) + 4u''(0)(\rho'(0))^2 - 2\rho''(0) = 0.$$

Substituting (1) and (2) in the above expression, we obtain

$$2g'(1)u(-2c) - 4g'(1)u(-c) - 2g''(1)u(-c) \\ -4(g'(1))^2u'(-c)u(-c) + 4(g'(1))^2u(-c) + 2(g'(1))^2u''(0)(u(-c))^2 = 0,$$

or equivalently,

$$u(-c)\left(g''(1) - 2(g'(1))^2 + 2g'(1)\right) - u(-2c)g'(1) \\ -(u(-c))^2u''(0)(g'(1))^2 + u'(-c)u(-c)2(g'(1))^2 = 0,$$

or, with $g'(1) \neq 0$,

$$u(-c)\left(\frac{g''(1)}{(g'(1))^2} - 2 + 2\frac{1}{g'(1)}\right) - u(-2c)\frac{1}{g'(1)} - (u(-c))^2u''(0) + 2u'(-c)u(-c) = 0. \quad (3)$$

Recall that $u(0) = 0$ and $u'(0) = 1$. To simplify notation, write

$$d := -c, \quad \delta := \frac{1}{g'(1)}, \quad \varepsilon := \frac{g''(1)}{(g'(1))^2} - 2 + 2\frac{1}{g'(1)}.$$

The MacLaurin expansion for $u(d)$ and its derivative can be written as

$$u(d) = d + \sum_{j=2}^{\infty} \frac{u^{(j)}(0)}{j!}d^j \qquad \text{and} \qquad u'(d) = 1 + \sum_{j=2}^{\infty} \frac{u^{(j)}(0)}{(j-1)!}d^{j-1}.$$

The equation (3) obtained above can be rewritten as

$$\varepsilon u(d) - \delta u(2d) - (u(d))^2\, u''(0) + 2u'(d)u(d) = 0, \quad \forall d. \quad (4)$$

All coefficients of d^j should be equal to zero on the lhs. There is no constant, and the coefficient of d^1 leads to

$$\varepsilon - 2\delta - 0 + 2 = \frac{g''(1)}{(g'(1))^2} = 0.$$

Therefore $g''(1) = 0$.

Equating the coefficient of d^2 to zero gives that we must have

$$u''(0)(1 - \delta) = 0.$$

Evidently, this holds if $\delta = 1$, which means that $g'(1) = 1$. But when $g'(1) = 1$ as well as $g''(1) = 0$, the differential equation (3) coincides with the one in Gerber (1974a, 1985), for which he proves that it is only satisfied if u is exponential or linear. Then by Heilpern (2003), g must be linear, hence, g is the identity function.

For the case $\delta = g'(1) \neq 1$, we will show that u must be linear, and then the result of the theorem follows again directly from Heilpern (2003). By the last equation, from $\delta \neq 1$ it follows that $u''(0) = 0$ must hold. To show that then $u^{(j)}(0) = 0$ also for all $j = 3, 4, \ldots,$ we use induction. So assume that $u^{(j)}(0) = 0$ holds for all $j = 2, \ldots, n-1$. Then the coefficient of d^n in the MacLaurin expansion of (4) can be written as

$$2(\delta - 1)\frac{u^{(n)}(0)}{n!} - \delta 2^n \frac{u^{(n)}(0)}{n!} + 0 + 2\left(\frac{u^{(n)}(0)}{n!} + 0\right) = \frac{u^{(n)}(0)}{n!}\left(2\delta - 2^n\delta\right).$$

Since $\delta > 0$ must hold, (4) holding implies that $u^{(n)}(0) = 0$, eventually for all $n = 2, 3, \ldots$ By Heilpern (2003), g must then be linear, which contradicts the stated assumption that $\delta = g'(1) \neq 1$. It means that $\delta = g'(1) \neq 1$ cannot hold. This completes the proof.

Remark 2.1 *We required that the function $u(\cdot)$ has a MacLaurin expansion, to be able to identify it from its derivatives at 0 only. This is not a trivial assumption; for example, defining $f(x) = \exp(-1/x^2)$ with $f(0) = 0$, it is easy to see that u and $u + f$ have the same derivatives at 0.*

3 Concluding Remarks

Goovaerts *et al.* (2004) prove a general representation result for additive risk measures. Their result entails that risk measures are normalized and additive, and respect exponential order (hence, are monotone) if and only if they are mixtures of exponential premiums; see also Gerber and Goovaerts (1981) and Goovaerts and Laeven (2008) for related results.

In contrast to Goovaerts *et al.* (2004), the additive ρ solving $\pi[X - \rho] = 0$ we find does not permit a (non-degenerate) mixture function in its representation. This can be explained by either one of the following two reasons: (i) The mixture function is not compatible with the comonotone independence axiom needed to axiomatize rank-dependent utility. (ii) The mixture function is not compatible with the iterativity property obtained here.

With respect to (ii) we note the following: Gerber (1974b) proved that a risk measure that satisfies a certain continuity condition is iterative if and only if it is a mean value principle. Furthermore, as is well-known, the mean value principle is additive (hence, translation invariant) if and only if it is an exponential premium. The mixture of exponential premiums of Goovaerts *et al.* (2004) satisfies the particular continuity condition of Gerber (1974b) and furthermore is additive. Hence, we conclude that it is iterative if and only if the mixture function is degenerate.

A second remark we want to make is that one may wonder whether a result similar to our main theorem holds for Choquet (1953-4) expected utility of Schmeidler (1989) or maxmin expected utility of Gilboa and Schmeidler (1989). But requiring additivity, or rather $\mathbb{P}$-additivity, in settings of *decision-making under uncertainty*, where the probability measure need not be known and given in advance, seems quite unnatural. For decision-making under risk, Choquet expected utility coincides with rank-dependent utility and maxmin expected utility with Von Neumann and Morgenstern (1944) expected utility. It means that for decision-making under risk, the main result contained in this note holds across the dominant decision-making paradigms.

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References

[1] Borch, Karl (1962). "Equilibrium in a reinsurance market," *Econometrica* 3, 424-444.

[2] Choquet, Gustave (1953-4). "Theory of capacities," *Annales de l'Institut Fourier* 5 (Grenoble), 131-295.

[3] Denuit, Michel, Jan Dhaene, Marc J. Goovaerts, Rob Kaas & Roger J.A. Laeven (2006). "Risk measurement with equivalent utility principles," In: Rüschendorf, Ludger (Ed.), *Risk Measures: General Aspects and Applications* (special issue), *Statistics and Decisions* 24 (1), 1-26.

[4] Gerber, Hans U. (1974a). "On additive premium calculation principles," *Astin Bulletin* 7, 215-222.

[5] Gerber, Hans U. (1974b). "On iterative premium calculation principles," *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* 74, 163-172.

[6] Gerber, Hans U. & Marc J. Goovaerts (1981). "On the representation of additive principles of premium calculation," *Scandinavian Actuarial Journal* 4, 221-227.

[7] Gerber, Hans U. (1985). "On additive principles of zero utility," *Insurance: Mathematics and Economics* 4, 249-251.

[8] Gilboa, Itzhak & David Schmeidler (1989). "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics* 18, 141-153.

[9] Goovaerts, Marc J., Rob Kaas, Roger J.A. Laeven & Qihe Tang (2004). "A comonotonic image of independence for additive risk measures," *Insurance: Mathematics and Economics* 35, 581-594.

[10] Goovaerts, Marc J. & Roger J.A. Laeven (2008). "Actuarial risk measures for financial derivative pricing," *Insurance: Mathematics and Economics* 42, 540-547.

[11] Goovaerts, Marc J., Rob Kaas & Roger J.A. Laeven (2010). "On risk measures and decisions in insurance and finance," *Insurance: Mathematics and Economics*, accepted for publication.

[12] Heilpern, Stanislaw (2003). "A rank-dependent generalization of zero utility principle," *Insurance: Mathematics and Economics* 33, 67-73.

[13] Quiggin, John (1982). "A theory of anticipated utility," *Journal of Economic Behaviour and Organization* 3, 323-343.

[14] Schmeidler, David (1989, first version 1982). "Subjective probability and expected utility without additivity," *Econometrica* 57, 571-587.

[15] Tversky, Amos & Daniel Kahneman (1992). "Advances in prospect theory: cumulative representation of uncertainty," *Journal of Risk and Uncertainty* 5, 297-323.

[16] Von Neumann, John & Oskar Morgenstern (1944). *Theory of Games and Economic Behavior*, Princeton: Princeton University Press. (Third edition 1953).

[17] Yaari, Menahem E. (1987). "The dual theory of choice under risk," *Econometrica* 55, 95-115.

Marc J. Goovaerts
Catholic University of Leuven
Dept. of Applied Economics
Naamsestraat 69
B-3000 Leuven, Belgium

and

University of Amsterdam
Dept. of Quantitative Economics
Roetersstraat 11
1018 WB Amsterdam, The Netherlands

Rob Kaas
University of Amsterdam
Dept. of Quantitative Economics
Roetersstraat 11
1018 WB Amsterdam, The Netherlands

Roger J.A. Laeven
Tilburg University and CentER
Dept. of Econometrics and Operations Research
P.O. Box 90153
5000 LE Tilburg, The Netherlands