Comonotonicity

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Abstract

Actuaries often have to determine the distribution function of a sum of random variables, such as the aggregate claims of an insurance portfolio over a certain future reference period. In general the random variables in an insurance portfolio are assumed to be mutually independent. Traditional risk theory offers plenty of techniques to deal with independent variables so this assumption is very convenient from a mathematical point of view. The assumption of mutual independence however doesn't always comply with reality, which may resolve in an underestimation of the total risk.

In this article we will discuss the comonotonic copula as a tool to deal with sums of dependent random variables whose marginal distributions are known, but with an unknown or complicated joint distribution. Considering comonotonic random vectors essentially reduces the multidimensional problem to a univariate one since then all components depend on the same variable.

Keywords: copula, correlation, maximum dependence, Fréchet bounds, ordering of risks, stop-loss premiums

1 Definition and characterizations

When dealing with stochastic orderings, actuarial risk theory generally focuses on single risks or sums of independent risks. Here risks denote non-negative random variables such as they occur in the individual and the collective model, see e.g. [8, 9]. Recently, with an eye on financial actuarial applications, the attention has shifted to sums $X_1 + X_2 + \cdots + X_n$ of random variables that may also have negative values. Moreover, their independence is no longer required. Only the marginal distributions are assumed to be fixed. A central result is that in this situation, the sum of the components $X_1 + X_2 + \cdots + X_n$ is the riskiest if the random variables X_i have a *comonotonic* copula.

We start by defining comonotonicity of a set of n-vectors in $\mathbb{R}^n$. A n-vector $(x_1, \ldots, x_n)$ will be denoted by $\underline{x}$. For two n-vectors $\underline{x}$ and $\underline{y}$, the notation $\underline{x} \leq \underline{y}$ will be used for the componentwise order which is defined by $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$.

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Definition 1 (Comonotonic set)
The set $A \subseteq \mathbb{R}^n$ is comonotonic if for any $\underline{x}$ and $\underline{y}$ in A, either $\underline{x} \leq \underline{y}$ or $\underline{y} \leq \underline{x}$ holds.

So, a set $A \subseteq \mathbb{R}^n$ is comonotonic if for any $\underline{x}$ and $\underline{y}$ in A, the inequality $x_i < y_i$ for some i, implies that $\underline{x} \leq \underline{y}$. As a comonotonic set is simultaneously non-decreasing in each component, it is also called a nondecreasing set, see [10]. Notice that any subset of a comonotonic set is also comonotonic.

Next we define a comonotonic random vector $\underline{X} = (X_1, \dots, X_n)$ through its support. A support of a random vector $\underline{X}$ is a set $A \subseteq \mathbb{R}^n$ for which $\text{Prob}[\underline{X} \in A] = 1$.

Definition 2 (Comonotonic random vector)
A random vector $\underline{X} = (X_1, \dots, X_n)$ is comonotonic if it has a comonotonic support.

From the definition, we can conclude that comonotonicity is a very strong positive dependency structure. Indeed, if $\underline{x}$ and $\underline{y}$ are elements of the (comonotonic) support of $\underline{X}$, i.e. $\underline{x}$ and $\underline{y}$ are possible outcomes of $\underline{X}$, then they must be ordered componentwise. This explains why the term comonotonic (common monotonic) is used.

In the following theorem, some equivalent characterizations are given for comonotonicity of a random vector.

Theorem 1 (Equivalent conditions for comonotonicity)
A random vector $\underline{X} = (X_1, X_2, \dots, X_n)$ is comonotonic if and only if one of the following equivalent conditions holds:

(a) $\underline{X}$ has a comonotonic support;

(b) $\underline{X}$ has a comonotonic copula, i.e. for all $\underline{x} = (x_1, x_2, \dots, x_n)$, we have

$$F_{\underline{X}}(\underline{x}) = \min \left\{ F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n) \right\}; \tag{1}$$

(c) For $U \sim$ Uniform(0,1), we have

$$\underline{X} \overset{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \dots, F_{X_n}^{-1}(U)); \tag{2}$$

(d) A random variable Z and non-decreasing functions f_i $(i = 1, \dots, n)$ exist such that

$$\underline{X} \overset{d}{=} (f_1(Z), f_2(Z), \dots, f_n(Z)). \tag{3}$$

From (1) we see that, in order to find the probability of all the outcomes of n comonotonic risks X_i being less than x_i $(i = 1, \dots, n)$ one simply takes the probability of the least likely of these n events. It is obvious that for any random vector $(X_1, \dots, X_n)$, not necessarily comonotonic, the following inequality holds:

$$\text{Prob}[X_1 \leq x_1, \dots, X_n \leq x_n] \leq \min \left\{ F_{X_1}(x_1), \dots, F_{X_n}(x_n) \right\}, \tag{4}$$

and since Hoeffding [7] and Fréchet [5] it is known that the function $\min\{F_{X_1}(x_1), \dots, F_{X_n}(x_n)\}$ is indeed the multivariate cdf of a random vector, i.c. $(F_{X_1}^{-1}(U), \dots, F_{X_n}^{-1}(U))$, which has the same marginal distributions as $(X_1, \dots, X_n)$. Inequality (4) states that in the class of all random vectors $(X_1, \ldots, X_n)$ with the same marginal distributions, the probability that all X_i simultaneously realize large values is maximized if the vector is comonotonic, suggesting that comonotonicity is indeed a very strong positive dependency structure. In the special case that all marginal distribution functions F_{X_i} are identical, we find from (2) that comonotonicity of $\underline{X}$ is equivalent to saying that $X_1 = X_2 = \ldots = X_n$ holds almost surely.

A standard way of modelling situations where individual random variables $X_1, \ldots, X_n$ are subject to the same external mechanism is to use a secondary mixing distribution. The uncertainty about the external mechanism is then described by a structure variable z, which is a realization of a random variable Z and acts as a (random) parameter of the distribution of $\underline{X}$. The aggregate claims can then be seen as a two-stage process: first, the external parameter $Z = z$ is drawn from the distribution function F_Z of z. The claim amount of each individual risk X_i is then obtained as a realization from the conditional distribution function of X_i given $Z = z$. A special type of such a mixing model is the case where given $Z = z$, the claim amounts X_i are degenerate on x_i, where the $x_i = x_i(z)$ are non-decreasing in z. This means that $(X_1, \ldots, X_n) \stackrel{d}{=} (f_1(Z), \ldots, f_n(Z))$ where all functions f_i are non-decreasing. Hence, $(X_1, \ldots, X_n)$ is comonotonic. Such a model is in a sense an extreme form of a mixing model, as in this case the external parameter $Z = z$ completely determines the aggregate claims.

If $U \sim$ Uniform(0,1), then also $1 - U \sim$ Uniform(0,1). This implies that comonotonicity of $\underline{X}$ can also be characterized by

$$\underline{X} \stackrel{d}{=} (F_{X_1}^{-1}(1-U), F_{X_2}^{-1}(1-U), \ldots, F_{X_n}^{-1}(1-U)).$$

Similarly, one can prove that $\underline{X}$ is comonotonic if and only if there exist a random variable Z and non-increasing functions f_i, $(i = 1, 2, \ldots, n)$, such that

$$\underline{X} \stackrel{d}{=} (f_1(Z), f_2(Z), \ldots, f_n(Z)).$$

Comonotonicity of a n-vector can also be characterized through pairwise comonotonicity.

Theorem 2 (Pairwise comonotonicity)
A random vector $\underline{X}$ is comonotonic if and only if the couples (X_i, X_j) are comonotonic for all i and j in $\{1, 2, \ldots, n\}$.

A comonotonic random couple can then be characterized using Pearson's correlation coefficient r (see [12]):

Theorem 3 (Comonotonicity and maximum correlation)
For any random vector (X_1, X_2) the following inequality holds:

$$r(X_1, X_2) \leq r\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right), \tag{5}$$

with strict inequalities when (X_1, X_2) is not comonotonic.

As a special case of (5), we find that $r(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)) \geq 0$ always holds. Note that the maximal correlation attainable does not equal 1 in general, see e.g. [4]. In [1] it is shown that other dependence measures such as Kendall's τ, Spearman's ρ and Gini's γ equal 1 (and thus are also maximal) if and only if the variables are comonotonic. Also note that a random vector (X_1, X_2) is comonotonic and has mutually independent components if and only if X_1 or X_2 is degenerate, see e.g. [8].

2 Sum of comonotonic random variables

In an insurance context, one is often interested in the distribution function of a sum of random variables. Such a sum appears for instance when considering the aggregate claims of an insurance portfolio over a certain reference period. In traditional risk theory, the individual risks of a portfolio are usually assumed to be mutually independent. This is very convenient from a mathematical point of view as the standard techniques for determining the distribution function of aggregate claims, such as Panjer's recursion, De Pril's recursion, convolution or moment based approximations, are based on the independence assumption. Moreover, in general the statistics gathered by the insurer only give information about the marginal distributions of the risks, not about their joint distribution, i.e. the way these risks are interrelated. The assumption of mutual independence however doesn't always comply with reality, which may resolve in an underestimation of the total risk. On the other hand, the mathematics for dependent variables is less tractable, except when the variables are comonotonic.

In the actuarial literature it is common practice to replace a random variable by a *less attractive* random variable which has a simpler structure, making it easier to determine its distribution function, see e.g. [6] and [9]. Performing the computations (of premiums, reserves and so on) with the less attractive random variable will be considered as a prudent strategy by a certain class of decision makers. From the theory on ordering of risks, we know that in case of stop-loss order this class consists of all risk-averse decision makers.

Definition 3 (Stop-loss order)
Consider two random variables X and Y. Then X precedes Y in the stop-loss order sense, written as $X \leq_{sl} Y$, if and only if X has lower stop-loss premiums than Y:

$$\mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+], \qquad d \in \mathbb{R},$$

with $(x-d)_+ = \max(x-d, 0)$.

Additionally requiring that the random variables have the same expected value leads to the so-called convex order.

Definition 4 (Convex order)
Consider two random variables X and Y. Then X precedes Y in the convex order sense, written as $X \leq_{cx} Y$, if and only if $\mathrm{E}[X] = \mathrm{E}[Y]$ and $\mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+]$ for all real d.

Now replacing the copula of a random vector by the comonotonic copula yields a less attractive sum in the convex order, see e.g. [2, 3].

Theorem 4 (Convex upper bound for a sum of random variables)
For any random vector $(X_1, X_2, \dots, X_n)$ we have

$$X_1 + X_2 + \ldots + X_n \leq_{cx} F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U) + \ldots + F_{X_n}^{-1}(U).$$

Furthermore, the distribution function and the stop-loss premiums of a sum of comonotonic random variables can be calculated very easily. Indeed, the inverse distribution function of the sum turns out to be equal to the sum of the inverse marginal distribution functions. For the stop-loss premiums we can formulate a similar phrase.

Theorem 5 (Inverse cdf of a sum of comonotonic random variables)
The inverse distribution function $F_{S^c}^{-1}$ of a sum S^c of comonotonic random variables with strictly increasing distribution functions $F_{X_1}, \dots, F_{X_n}$ is given by

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p), \qquad 0 < p < 1. \tag{6}$$

Theorem 6 (Stop-loss premiums of sum of comonotonic random variables)
The stop-loss premiums of the sum S^c of the components of the comonotonic random vector with strictly increasing distribution functions $F_{X_1}, \dots, F_{X_n}$ are given by

$$\mathrm{E}[(S^c - d)_+] = \sum_{i=1}^{n} \mathrm{E}\left[(X_i - F_{X_i}^{-1}(F_{S^c}(d))_+\right],$$

for all $d \in \mathbb{R}$.

From (6) we can derive the following property (see [11]): if the random variables X_i can be written as a linear combination of the same random variables $Y_1, \dots, Y_m$, i.e.,

$$X_i \stackrel{d}{=} a_{i,1}Y_1 + \cdots + a_{i,m}Y_m,$$

then their comonotonic sum can also be written as a linear combination of $Y_1, \dots, Y_m$. Assume for instance that the random variables X_i are Pareto(α, β_i) distributed with fixed first parameter, i.e.

$$F_{X_i}(x) = 1 - \left(\frac{\beta_i}{x}\right)^{\alpha}, \qquad \alpha > 0,\ x > \beta_i > 0,$$

or $X_i \stackrel{d}{=} \beta_i X$, with $X \sim$ Pareto(α,1), then the comonotonic sum is also Pareto distributed, with parameters α and $\beta = \sum_{i=1}^{n} \beta_i$. Other examples of such distributions are Exponential, Normal, Rayleigh, Gumbel, Gamma (with fixed first parameter), Inverse Gaussian (with fixed first parameter), Exponential-Inverse Gaussian, etc.

Besides these interesting statistical properties, the concept of comonotonicity has several actuarial and financial applications such as determining provisions for future payment obligations or bounding the price of Asian options, see [3].

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